# $LQ$ FEEDBACK FORMULATION FOR $H_\infty$ OUTPUT FEEDBACK CONTROL

by

Anantha Karthikeyan

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Dedicated to my family

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# Abstract

In this thesis we present a simple, unified formula for discrete and continuous-time $H_\infty$ "all-solutions" controllers. By observing a "cost" equivalence between the standard $H_\infty$ control problem and a certain $LQ$ optimal regulator problem, an elegant controller structure reminiscent of an $LQG$ optimal controller is developed. Our choice of notation also simplifies the derivation and existence conditions considerably, whereby all unnecessary assumptions on plant state-space matrices and "loop-shifting" transformations are eliminated. Additionally, with our focus entirely on input-output weighting "cost functions" this derivation offers a "behavioral" theory interpretation for all solutions of a standard $H_\infty$ control problem.

In this thesis we also present a simplified matrix pencil formula for solving the $\mathrm{H}_\infty$ control problem for the case $0 \leq \bar{\sigma}(D_{11}) \leq \gamma$. This formula is useful in developing a more numerically robust algorithm in $\mathrm{H}_\infty$ control. A significant feature of this formula is that each element of the pencils is expressed directly in terms of the original descriptor-form state space matrices of the plant and even pencil eigenspaces computed using a numerically robust even pencil algorithm. There are no data-corrupting numerical operations required to form any of the matrices that appear in our "all-solutions" controller formula.

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# Chapter 1

# Introduction

The idea of applying $LQ$ game theory to robust control problems can be traced to the 1967 paper by Medanic [Medanic, 1967]. Building upon this idea, Mageirou and Ho developed optimal small gain feedback theory [Mageirou and Ho, 1977] in 1977. After a brief hiatus, the idea re-surfaced in the form of $LQ$ 'completion of squares' identity in the seminal work of Doyle et al. [Doyle et al., 1989]. By then, techniques to simplify $H_\infty$ theory via "loop shifting" concept were also in place [Safonov et al., 1989]. Finally a characterization of all solutions to the four block general distance problem was presented by Limebeer et al. [Limebeer et al., 1988] in 1991. In these works, restrictive conditions on the state-space $A, B, C, D$-matrices were used to keep the equations simple, and then the case of general $A, B, C, D$-matrices is handled via a sequence of additional loop-shifting transformations [Safonov et al., 1989] which add considerably to the complexity of the derivations and to the controller formulae. An LQ game theoretic formulation [Medanic, 1967, Mageirou and Ho, 1977, Doyle et al., 1989] of continuous-time $H_\infty$ output-feedback control was presented in [Karthikeyan and Safonov, 2010], along with an "all-solutions" characterization of the controller. The result of this paper showed that the seminal works of [Doyle et al., 1989] can be simplified in the case of general $A, B, C, D$-matrices without any additional "loop-shifting" transformations [Safonov et al., 1989], thereby significantly reducing the complexity of the derivations, controller formulae and existence conditions.

As for the case of discrete-time $H_\infty$ control, an indirect approach has been to use bilinear transforms along with the continuous-time results to find a suitable controller. To circumvent the use of bilinear transforms many techniques have been proposed, of which, significant contributions were made by [Limebeer et al., 1989, Green and Limebeer, 1995, Iglesias and Glover, 1991, Ionescu et al., 1999, Stoorvogel et al., 1994] and [Petkov et al., 1999]. Furthermore, in [Karthikeyan and Safonov, 2009] the author showed the existence of a deeper connection between the discrete and continuous-time cases, although an “all-solutions” characterization of the controller was not presented here.

The motivation for the $LQ$ feedback formulation results in this thesis are therefore two-fold. First we wish to demonstrate that the derivation for the continuous-time case has almost a one-to-one mapping to the discrete-time “all-solutions” case, which calls for a unified theory and software implementation. Second, we wish to show that this “$LQ$ cost function” approach completely eliminates numerically corrupting operations such as “loop-shifting” transformation and bilinear transforms from $H_\infty$ controller derivations.

When it comes to the question of numerically robust methods in $\mathrm{H}_\infty$ control we see that techniques like [Benner et al., 2007] have been developed based on gamma iteration and a novel extended matrix pencil formulation of the state space solution of the sub-optimal $H_\infty$ control problem. This approach is based on solving even generalized eigenproblems instead of Riccati equations and unstructured matrix pencils. Such methods avoid potentially error causing matrix algebra involving summation and inversion of ill-conditioned matrices which would otherwise be encountered while constructing the Riccati equation or Hamiltonian matrices used for solving the gamma iteration problem. The enhanced numerical robustness in this method comes from preserving the spectral symmetries which are inherent in the structure of the problem. Furthermore, these methods are found to be useful even if the pencil has eigenvalues on the imaginary axis. However, the problem of bringing in numerical robustness into the controller formula was not addressed in [Benner et al., 2007]. Therefore, in this thesis we present another simplified "all-solutions" formula using inverse free matrix pencils, where each element of the controller is expressed in terms of the original descriptor form state-space representation of the plant and even pencil eigenspaces computed using the structure preserving algorithm of [Benner et al., 2007]. This implies reduced numerical manipulation and correspondingly data corruption while solving the $H_\infty$ control problem.

# Chapter 2

# Preliminaries

## 2.1 Notation and Terminology

1. The operator '$\delta$' is defined as

$$\delta = \begin{cases} z, & \text{(Discrete-time)} \\ s, & \text{(Continuous-time)} \end{cases}$$

   where '$z$' is the forward time-shift operator and '$s$' is the differentiation operator.

2. We consider a real linear time-invariant system $G$ having state-space system matrix

$$S(G) \overset{\text{ss}}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{2.1.1}$$

   The state-space system $S(G)$ has transfer function

$$G(\delta) = C(\delta I - A)^{-1}B + D$$

3. We denote by $\mathrm{RH}_\infty$ the set of real LTI transfer function matrices which are stable and proper.

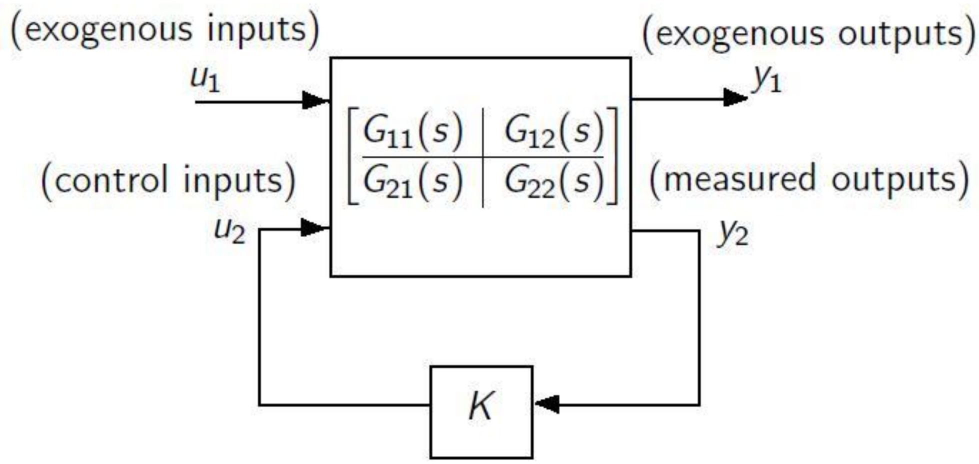


Figure 2.1: General control configuration

4. For $G \in \mathrm{RH}_\infty$, we define the $H_\infty$ norm as (e.g., [Skogestad and Postlethwaite, 2005]),

$$\| G(\delta) \|_\infty \triangleq \begin{cases} \sup_\omega \bar{\sigma}(G(\delta)|_{\delta=e^{j\omega}}), & \delta = z \\ \sup_\omega \bar{\sigma}(G(\delta)|_{\delta=j\omega}), & \delta = s \end{cases}$$

where $\bar{\sigma}$ denotes the greatest singular value of a matrix.

5. Given a plant $G$ and controller $K$ partitioned compatibly as follows

$$G = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right]$$
$$K = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right].$$

If $G_{22}K_{11}$ is well defined and square, and $I - G_{22}K_{11}$ is invertible, then $T =$

$\mathrm{lft}(G, K)$ forms the *Redheffer star product* or linear fractional transformation with respect to this partition [Matlab, 2010a], [Redheffer, 1960] (see Fig.2.1).

6. $\ell_2[0, \infty)$ denotes the Hilbert space of vector valued functions defined on $[0, \infty)$ with inner product

$$\langle f, g \rangle = \sum_{k=0}^{\infty} (f(k)^T g(k)). \tag{2.1.2}$$

7. $L_2[0, \infty)$ denotes the Hilbert space of vector valued functions defined on $[0, \infty)$ with inner product

$$\langle f, g \rangle = \int_0^{\infty} (f(t)^T g(t)) \, dt. \tag{2.1.3}$$

8. By the notation $\mathfrak{L}_2[0, \infty)$ we refer to the Hilbert space $\ell_2[0, \infty)$ in the discrete-time case or the space $L_2[0, \infty)$ in the continuous-time case.

9. Given a discrete or continuous-time signal $u$ we define the $\mathfrak{L}_2$ norm

$$\| u \|_{\mathfrak{L}_2}^2 \triangleq \begin{cases} \| u \|_{\ell_2[0,\infty)}^2, & \text{(Discrete-time)} \\ \| u \|_{L_2[0,\infty)}^2, & \text{(Continuous-time)} \end{cases}$$

10. CoS is used as an abbreviation for the phrase: “completion of squares”.

## 2.2 Definitions

### 2.2.1 Schur Decomposition

**Definition 1** (Schur Decomposition [Skogestad and Postlethwaite, 2005]). *Given a matrix $A$ partitioned as follows*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.2.1}$$

*if $A_{22}$ is non-singular then $A$ has the decomposition*

$$A = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} Y & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix} \tag{2.2.2}$$

*where $Y = A_{11} - A_{12}A_{22}^{-1}A_{21}$.*

### 2.2.2 Schur $J$-factorization

**Definition 2** (Schur $J$-factorization). *Let $A$ be a symmetric matrix partitioned as in equation (2.2.1). If $A$ has a Schur decomposition (2.2.2) with $A_{22} > 0$ and $Y < 0$, then for a given value of $\gamma$ there exists a Schur $J$-factorization*

$$A = \begin{bmatrix} I & S_{21}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & S_1^T \\ S_2^T & 0 \end{bmatrix} J \begin{bmatrix} 0 & S_2 \\ S_1 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ S_{21} & I \end{bmatrix} \tag{2.2.3}$$

Table 2.1: `MATLAB` script for `SJF` function

```
function [S1,S2,S21,exist]=SJF(A,a1,gamma)
A11 =A(1:a1,1:a1); % partition A
A12 =A(1:a1,a1+1:end); A21 =A12';
A22 =A(a1+1:end,a1+1:end);
Y = A11-A12*(A22\A12');
% check existence and compute Schur J-Factors
exist1=true((eig(A22))'>(zeros(size(eig(A22))))')
exist2=true((eig(Y))'<(zeros(size(eig(Y))))')
if and(exist1,exist2)
   S1 =sqrt(-Y)./gamma;
   S2 =sqrt(A22); S21 =(A22\A12');
   exist=true;
else
   S1=[];S2=[];S21=[];
   exist=false;
end
```

*where the matrix $J$ and Schur $J$-factors $S_1, S_2, S_{21}$ are given by equations (2.2.4-2.2.7).*

$$J \triangleq \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}, \tag{2.2.4}$$

$$S_1 \triangleq \gamma^{-1}(-Y)^{\frac{1}{2}}, \tag{2.2.5}$$

$$S_2 \triangleq (A_{22})^{\frac{1}{2}}, \tag{2.2.6}$$

$$S_{21} \triangleq A_{22}^{-1} A_{12}^T. \tag{2.2.7}$$

**The $\mathrm{SJF}(A, \gamma)$ function**

**Definition 3.** $(\mathrm{SJF}(A, \gamma)$ *The notation* $\mathrm{SJF}(A, \gamma)$ *denotes a function that computes the Schur $J$-factorization of (2.2.3) whenever such a factorization exists.*

**Remark 1.** *Table (2.1) gives a* `MATLAB` *script which can be used to find Schur-$J$ factors of $A$ when they exist.*

### 2.2.3 Matrix Pencil

**Definition 4.** *Matrix pencils are order one polynomial matrices of the form* $M(s) = -sM_\alpha + M_\beta$*, where* $M_\alpha$*,* $M_\beta$ *are matrices. A pencil* $M(s)$ *is said the be even if the matrices* $M_\alpha$*,* $M_\beta$ *are real and* $M(s) = M^T(-s)$*. A generalized eigenvector* $\lambda_0$ *and zero at* $s_0$ *can be evaluated if* $M(s_0)\lambda_0$*=0 and* $M(s)\lambda_0 \neq 0$ *for all* $s \neq s_0$*.*

## 2.3 Problem Statements

### 2.3.1 Standard $H_\infty$ Control Problem

The standard $H_\infty$ control problem concerns finding a controller $K$ for a plant $G$ defined by the state equations

$$\begin{bmatrix} \delta x \\ y_1 \\ y_2 \end{bmatrix} = S(G) \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix}, \qquad (2.3.1)$$

where

$$S(G) \overset{\text{ss}}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \qquad (2.3.2)$$

$D_{22} \in \mathbb{R}^{m_2 \times r_2}$, $D_{11} \in \mathbb{R}^{m_1 \times r_1}$, $m = m_1 + m_2$, $r = r_1 + r_2$.

**Definition 5.** $\Gamma$ *is the set of all* $\gamma \geq 0$ *for which an internally stabilizing controller exists and*

$$T_{y_1 u_1} \triangleq \mathrm{lft}(G, K)$$

*satisfies*

$$\|T_{y_1 u_1}\|_\infty < \gamma. \qquad (2.3.3)$$

**Problem 1** (Standard $H_\infty$ Control Problem)**.** *[Doyle et al., 1989]*
*Given* $\gamma \geq 0$*, the standard* $H_\infty$ problem *is to determine if* $\gamma \in \Gamma$ *and, if so, to compute a realization for* $K$. □

### 2.3.2 Optimal $H_\infty$ Control Problem

**Problem 2** (Optimal $H_\infty$ Control)**.**

*The* optimal $H_\infty$ problem *is to compute the infimum of* $\Gamma$

$$\gamma_{opt} \triangleq \min_{\Gamma} \gamma$$

*and a corresponding controller (e.g.,[Benner et al., 2007]).* □

In practice, the solution to optimal Problem 2 is computed via the $\gamma$-iteration algorithm (e.g., [Benner et al., 2007]) in which a convergent sequence upper and lower bounds on $\gamma_{opt}$ is computed via iterative solution of Problem 1. Although, our main concern here is the solution of the standard Problem 1.

### 2.3.3 $H_\infty$ Full-Information and Full-Control Problems

**Problem 3** ($H_\infty$ Full-Information Feedback)**.**

*We refer to Problem 1 as the $H_\infty$ Full-information problem when the plant $G$ is replaced by the corresponding Full-Information plant $G_{FI}$*

$$S(G_{FI}) \overset{ss}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ \begin{bmatrix} I_n \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I_{r_1} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]. \tag{2.3.4}$$

□

**Problem 4** ($H_\infty$ Full-Control Feedback)**.**

*We refer to Problem 1 as the $H_\infty$ Full-control problem when the plant $G$ is replaced by the corresponding Full-Control plant $G_{FC}$*

$$S(G_{FC}) \overset{ss}{=} \left[\begin{array}{c|cc} A & B_1 & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline C_1 & D_{11} & \begin{bmatrix} 0 & I \end{bmatrix} \\ C_2 & D_{21} & \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array}\right]. \tag{2.3.5}$$

□

### 2.3.4 $H_\infty$ Disturbance Feed-forward and Output Estimation Problems

**Problem 5** ($H_\infty$ Disturbance Feed-forward)**.**

*We refer to Problem 1 as the $H_\infty$ Disturbance Feed-forward problem when the plant $G$ is replaced by the corresponding Disturbance Feed-forward plant $G_{DF}$*

$$S(G_{DF}) \overset{ss}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & 0 \end{array}\right]. \tag{2.3.6}$$

*To motivate the name disturbance feed-forward consider the special case with $C_2 = 0$. Then there is no feedback and the measurement is exactly $u_1$.* □

**Problem 6** ($H_\infty$ Output Estimation)**.**

*We refer to Problem 1 as the $H_\infty$ Output Estimation problem when the plant $G$ is replaced by the corresponding Output Estimation plant $G_{OE}$*

$$S(G_{OE}) \stackrel{ss}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & I \\ C_2 & D_{21} & 0 \end{array}\right]. \tag{2.3.7}$$

□

**Remark 2.** *The Output Estimation problem is dual to the Disturbance Feed-forward problem just as Full-Control was dual to the Full-Information control problem.*

### 2.3.5 $LQ$ State-feedback Control Problem

**Problem 7** ($LQ$ State-feedback Control Problem)**.**
*Given a state space system (2.1.1) and cost matrices $Q, R, N$, the Linear Quadratic (LQ) state-feedback control problem concerns finding a control law $u = Fx$ that minimizes the corresponding "output-weighting" cost function*

$$\Psi = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} Q & N \\ N^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle \tag{2.3.8}$$

*where $Q$ is any symmetric matrix.*
*Since $y = Cx + Du$, we may re-write the above cost in standard LQR optimal control framework as follows (see for e.g.,[Lancaster and Rodman, 1995, Matlab, 2008])*

$$\Psi = \left\langle \begin{bmatrix} x \\ u \end{bmatrix}, \begin{bmatrix} Q_x & N_x \\ N_x^T & R_x \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle \tag{2.3.9}$$

Table 2.2: `MATLAB` script for `lqy` function

```
function [F,Ru,exist]=lqy(G,Q,R,N)
[A,B,C,D,T]=ssdata(G);
Qx = C'*Q*C;
Rx = R + D'*Q*D + N'*D + D'*N;
Nx = C'*(Q*D + N);
if (T==0)   % continuous-time case
   [P,L,F] = care(A,B,Qx,Rx,Nx);
         Ru = Rx;
else        % discrete-time case
   [P,L,F] = dare(A,B,Qx,Rx,Nx);
         Ru = Rx + B'*P*B;
end
exist= (¬isequal(F,[]))&&(¬isequal(P,[]));
F=-F; % change sign for positive feedback u=Fx
```

*where,*

$$\begin{bmatrix} Q_x & N_x \\ N_x^T & R_x \end{bmatrix} \triangleq \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} Q & N \\ N & R \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}.$$

□

**The** lqy **function**

**Definition 6.** (lqy) *The notation* $\mathrm{lqy}(G, Q, R, N)$ *denotes a function that finds the unique stabilizing feedback* $F$ *that solves problem 7 for a given plant* $G$ *and choice of* $Q, R, N$*, whenever such an* $F$ *exists.*

□

**Remark 3.** *A* `MATLAB` *script for such a function is given in Table (2.2).*

# Chapter 3

# Background

## 3.1 Completion of Squares (*CoS*) Identities

### 3.1.1 $LQ$ Completion of Squares

**Lemma 3.1.1** ($LQ$ Completion of Squares Identity)**.** *[Brockett, 1970, Lancaster and Rodman, 1995]*

*For a given plant $G$ and cost matrices $Q, R, N$, if there exists a stabilizing feedback $F$ that solves problem 7, then, for all $u \in \mathfrak{L}_2[0,\infty)$, $u^* = Fx$ and initial condition $x(0) = 0$, it holds that*

$$\Psi \quad = \langle (u - u^*), R_u(u - u^*) \rangle, \tag{3.1.1}$$

*where*

$$R_u \quad = \begin{cases} R_x + B^T P B, & \textit{(Discrete-time)} \\ R_x, & \textit{(Continuous-time).} \end{cases} \tag{3.1.2}$$

*$P$ in equation (3.1.2) is the unique stabilizing solution of the discrete algebraic Riccati equation*

$$P = A^T P A + Q_x - (N_x^T + B^T P A)^T R_P^{-1} (N_x^T + B^T P A). \tag{3.1.3}$$

*Proof.* For the continuous-time case [Karthikeyan and Safonov, 2010], it is known [Brockett, 1970, Lancaster and Rodman, 1995] that for all $u(t) \in \mathbb{R}^m$ the cost $J_\tau$ at time $\tau$ with control $u$ and initial state $x_0$ can be represented in terms of any solution $P$ to the following Riccati equation

$$A^T P + PA - (PB + N_x){R_x}^{-1}(PB + N_x)^T + Q_x = 0 \tag{3.1.4}$$

and feedback gain matrix

$$F = -{R_x}^{-1}({N_x}^T + B^T P). \tag{3.1.5}$$

Now, if $P$ is any symmetric solution to the Riccati equation 3.1.4, then for any $u \in U$, $u^* = Fx$ and initial condition $x_0$,

$$J_\tau \quad = {x_0}^T P x_0 - {x_\tau}^T P x_\tau + \int_0^\tau (u - u^*)^T R_x (u - u^*)\; dt.$$

where

$$u^* = Fx.$$

Moreover, if $u \in L_2[0, \infty)$, $x_0 = 0$, then $\lim_{\tau \to \infty} x_\tau = 0$ and $\lim_{\tau \to \infty} J_\tau = J$. Therefore,

$$J = \int_0^\infty (u - u^*)^T R_x (u - u^*)\; dt. \tag{3.1.6}$$

For discrete-time case [Karthikeyan and Safonov, 2009] it is known [Brockett, 1970, Lancaster and Rodman, 1995] that for all $u(k) \in \mathbb{R}^m$ the cost $J$ at $k$ with control $u$

and initial state $x(0)$ can be represented in terms of any solution $P$ to the following Riccati equation

$$P = A^T PA + \tilde{Q} - (\tilde{S}^T + B^T PA)^T (R_P)^{-1} (\tilde{S}^T + B^T PA) \tag{3.1.7}$$

and feedback gain matrix

$$F = -R_P{}^{-1}(\tilde{S}^T + B^T PA) \tag{3.1.8}$$

where $R_P = \tilde{R} + B^T PB$. Now, if $P$ is any symmetric solution to the Riccati equation stated above, then for any $u \in U$ and initial condition $x(0)$

$$J_K = -x(K)^T Px(K) + x(0)^T Px(0) + \sum_{k=0}^{K} (u(k) - u(k)^*)^T R_P (u(k) - u(k)^*) \tag{3.1.9}$$

where

$$u(k)^* = Fx(k) \tag{3.1.10}$$

Moreover, if $u \in \ell_2[0, \infty)$, $x(0) = 0$ and $\lim_{K\to\infty} x(K) = 0$, then

$$\lim_{K\to\infty} J_K = \sum_{k=0}^{\infty} (u(k) - u(k)^*)^T R_P (u(k) - u(k)^*) \tag{3.1.11}$$

□

□

**Remark 4.** *It must be stressed that the "completion of squares" matrix $R_u$ in equation (3.1.6) is the only point of disparity between the continuous and discrete-time case.*

### 3.1.2 $LQ$ Interpretation of $H_\infty$

It is a well-known consequence of Parseval's theorem that the $H_\infty$ control objective $\|T_{y_1 u_1}\|_\infty < \gamma$ can be expressed equivalently in the time-domain in terms of an inequality satisfied by an $LQ$ cost function.

**Lemma 3.1.2** ($LQ$ cost interpretation of $H_\infty$)**.** *[Mageirou and Ho, 1977, Doyle et al., 1989]*

*For a given plant $G$ and controller $K$*

$$\|T_{y_1 u_1}\|_\infty < \gamma$$

*if and only if*

$$\Psi_{H_\infty} < 0 \text{ for all } u_1 \in \mathfrak{L}_{2[0,\infty)}, \tag{3.1.12}$$

*where*

$$\Psi_{H_\infty} = \|y_1\|^2_{\mathfrak{L}_2} - \gamma^2 \|u_1\|^2_{\mathfrak{L}_2}. \tag{3.1.13}$$

□

By defining cost matrices $Q_c, R_c, N_c$ as follows

$$Q_c \triangleq \begin{bmatrix} I_{m_1 \times m_1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{m \times m}, \tag{3.1.14}$$

$$R_c \triangleq \begin{bmatrix} -\gamma^2 I_{r_1 \times r_1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{r \times r}, \tag{3.1.15}$$

$$N_c \triangleq 0 \in R^{m \times r}. \tag{3.1.16}$$

equation (3.1.13) can be expressed as

$$\Psi_{H_\infty} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} Q_c & N_c \\ N_c^T & R_c \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \right\rangle. \tag{3.1.17}$$

Comparing cost functions $\Psi_{H_\infty}$ (3.1.17) and $\Psi$ (2.3.8), we notice that $\Psi_{H_\infty}$ is a special case of $\Psi$ where $Q = Q_c, R = R_c, N = N_c$. Note that we do *not* require $R_c > 0$.

**Lemma 3.1.3** ($H_\infty$ CoS Identity)**.**

*For a given plant $G$ and cost matrices $Q_c, R_c, N_c$ the $H_\infty$ "completion of squares" identity (3.1.18) holds for all $y_1, \bar{y}_1, u_1, \bar{u}_1, u_2, x \in \mathfrak{L}_2$ if and only if there exists a stabilizing feedback $F$ solving problem 7 and the corresponding LQ "completion of squares" matrix $R_u$ given by (3.1.2) has a Schur $J$-factorization (2.2.3).*

$$\|y_1\|^2_{\mathfrak{L}_2} - \gamma^2 \|u_1\|^2_{\mathfrak{L}_2} = \|\bar{y}_1\|^2_{\mathfrak{L}_2} - \gamma^2 \|\bar{u}_1\|^2_{\mathfrak{L}_2} \tag{3.1.18}$$

*where*

$$\begin{bmatrix} \bar{y}_1 \\ \bar{u}_1 \end{bmatrix} \triangleq \begin{bmatrix} 0 & S_{u_2} \\ S_{u_1} & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ S_{u_{21}} & I \end{bmatrix} \begin{bmatrix} u_1 - u_1^* \\ u_2 - u_2^* \end{bmatrix} \tag{3.1.19}$$

$$\begin{bmatrix} u_1^* \\ u_2^* \end{bmatrix} \triangleq \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} x \tag{3.1.20}$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \triangleq F. \tag{3.1.21}$$

$$S_{u_1} \triangleq \gamma^{-1}(R_{u_{12}} R_{u_{22}}^{-1} R_{u_{12}}^T - R_{u_{11}})^{\frac{1}{2}} \tag{3.1.22}$$

$$S_{u_2} \triangleq (R_{u_{22}})^{\frac{1}{2}} \tag{3.1.23}$$

$$S_{u_{21}} \triangleq R_{u_{22}}^{-1} R_{u_{12}}^T \tag{3.1.24}$$

*Note that by taking $A = R_u$ in equation (2.2.3), the Schur $J$-factors $S_{u_1}, S_{u_2}, S_{u_{21}}$ were derived above using equations (3.1.22-2.2.7).*

*Proof.* Specializing Lemma 3.1.1 to the case of $H_\infty$ cost matrices $Q = Q_c$, $R = R_c$, $N = N_c$, equation (3.1.6) becomes

$$\Psi_{H_\infty} = \langle (u - u^*), R_u(u - u^*) \rangle. \tag{3.1.25}$$

Using a Schur $J$-factorization of $R_u$ (2.2.3) in equation (3.1.25)

$$\Psi_{H_\infty} = \|\bar{y}_1\|_{\mathfrak{L}_2}^2 - \gamma^2 \|\bar{u}_1\|_{\mathfrak{L}_2}^2. \tag{3.1.26}$$

□

## 3.2 Squared-Down Plants

### 3.2.1 Squaring-Down $D_{12}$

**Lemma 3.2.1** (Squaring-Down $D_{12}$).

*Consider the squared-down plant $\bar{G}$ with a state-space representation*

$$S(\bar{G}) \overset{ss}{=} \left[\begin{array}{c|cc} \bar{A} & B_1 S_{u_1}^{-1} & B_2 \\ \hline -S_{u_2}F_2 & S_{u_2}S_{u_{21}}S_{u_1}^{-1} & S_{u_2} \\ \bar{C}_2 & D_{21}S_{u_1}^{-1} & D_{22} \end{array}\right] \tag{3.2.1}$$

*where $\bar{A} = A + B_1F_1$, $\bar{C}_2 = C_2 + D_{21}F_1$.*

*Under the conditions of Lemma 3.1.3, a feedback $u_2 = Ky_2$ that solves the $H_\infty$ control problem for the squared-down plant $\bar{G}$ also solves the $H_\infty$ control problem for the plant $G$, provided that it stabilizes $G$. Conversely, a feedback $u_2 = Ky_2$ that solves the $H_\infty$ control problem for the plant $G$ also solves the $H_\infty$ control problem for the squared-down plant $\bar{G}$, provided that it stabilizes $\bar{G}$.*

*Proof.* Given a plant $G$ with state-space equations (2.3.1), the equations for a squared down plant are obtained by substituting $\bar{u}_1 \triangleq S_{u_1}(u_1 - u_1^*)$, replacing the $y_1$ output by $\bar{y}_1 \triangleq S_{u_2}((u_2 - u_2^*) + S_{u_{21}}(u_1 - u_1^*))$ and regrouping terms. Since by hypothesis both $\bar{G}$ and $G$ are stabilized by $K$, the closed-loop response signals $y_1, y_2, u_2, x, \bar{y}_1, \bar{u}_2$ are in $\mathfrak{L}_2$ for all $u_1, \bar{u}_2$ in $\mathfrak{L}_2$ . Hence, by Lemma 3.1.3,

$$\|y_1\|_{\mathfrak{L}_2}^2 - \gamma^2\|u_1\|_{\mathfrak{L}_2}^2 = \|\bar{y}_1\|_{\mathfrak{L}_2}^2 - \gamma^2\|\bar{u}_1\|_{\mathfrak{L}_2}^2 \tag{3.2.2}$$

and the result follows immediately via Parseval's theorem. □

-0.1in By defining the cost matrices $Q_o, R_o, N_o$ as given below

$$Q_o \triangleq \begin{bmatrix} I_{r_1\times r_1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{r\times r} \tag{3.2.3}$$

$$R_o \triangleq \begin{bmatrix} -\gamma^2 I_{m_1\times m_1} & 0 \\ 0 & 0 \end{bmatrix} \in R^{m\times m} \tag{3.2.4}$$

$$N_o \triangleq 0 \in R^{r\times m} \tag{3.2.5}$$

the following dual result is immediate.

### 3.2.2 Squaring-Down $D_{21}$

**Lemma 3.2.2** (Squaring-Down $D_{21}$)**.**
*Given a plant $G$ and cost matrices $Q_o, R_o, N_o$, if for $G^T$ there exists a stabilizing feedback $H^T$ solving problem 7 such that the corresponding* CoS *matrix $R_y$ has Schur J-factors $S_{y_1}, S_{y_2}, S_{y_{21}}$, then a feedback $u_2 = Ky_2$ that solves the standard $H_\infty$ control Problem 1 for the following squared-down plant $\tilde{G}$*

$$S(\tilde{G}) \overset{ss}{=} \left[\begin{array}{c|cc} \tilde{A} & -H_2 S_{y_2} & \tilde{B}_2 \\ \hline S_{y_1}^{-1} C_1 & S_{y_1}^{-1} S_{y_{21}}^T S_{y_2} & S_{y_1}^{-1} D_{12} \\ C_2 & S_{y_2} & D_{22} \end{array}\right] \tag{3.2.6}$$

*where $\tilde{A} = A + H_1 C_1$, $\tilde{B}_2 = B_2 + H_1 D_{12}$, also solves the $H_\infty$ control problem for the plant G, provided that it stabilizes G. Conversely, a feedback $u_2 = Ky_2$ that solves the $H_\infty$ control problem for the plant $G$ also solves the $H_\infty$ control problem for the* *squared-down plant $\tilde{G}$, provided that it stabilizes $\tilde{G}$.*

*Proof.* Since $\| G \|_\infty = \| G^T \|_\infty$, the result follows directly by transposing $G$, applying Lemma 3.2.1 and finally transposing the resultant equations. □

# Chapter 4

# $H_\infty$ Full-Information and Full-Control

## 4.1 $H_\infty$ Full-Information Feedback

**Theorem 4.1.1** ($H_\infty$ Full-Information Feedback)**.**

*A solution to Problem 3 exists if and only if the following conditions hold:*

*(i) There exists a full-state feedback $F$ and CoS matrix $R_u$ such that the $H_\infty$ CoS identity (3.1.18) holds for the full-information plant $G_{FI}$ (2.3.4) and cost matrices $Q_c, R_c, N_c$.*

*(ii) The matrix $A + B_2\left(\left[\begin{array}{cc} S_{u_{21}} & I_{r_2} \end{array}\right]\right) F$ is Hurwitz.*

*Furthermore if a solution exists, then for $X \in RH_\infty$, $\|X\|_\infty < \gamma$, all solutions are given by*

$$u_2 = \operatorname{lft}(K_{FI}, X)\left[\begin{array}{c} x \\ u_1 \end{array}\right], \tag{4.1.1}$$

*using the "central full-information controller"*

$$K_{FI} = \operatorname{lft}(M_{FI}, S_{FI}), \tag{4.1.2}$$

*where*

$$
M_{FI} = \left[\begin{array}{cc|c} 0 & 0 & I \\ \hline F & \begin{bmatrix} -I \\ 0 \end{bmatrix} & 0 \end{array}\right], \tag{4.1.3}
$$

$$
S_{FI} = \left[\begin{array}{cc|c} S_{u_{21}} & I & S_{u_2}^{-1} \\ \hline -S_{u_1} & 0 & 0 \end{array}\right]. \tag{4.1.4}
$$

*Proof.* See Appendix. □

**Remark 5.** *The condition (ii) in Theorem 4.1.1 is equivalent to the requirement that the solution $P$ to the algebraic Riccati equation for the corresponding $LQ$ problem be positive semidefinite [Safonov et al., 1989, Thm 4]. From a computational standpoint, the Hurwitz condition (ii) in Theorem 4.1.1 is preferable. This is in part because the Hurwitz condition directly checks that the feedback gain $F$ is stabilizing. But, more importantly, condition (ii) circumvents the numerical sensitivity issues that would arise in attempting to distinguish an indefinite Riccati equation solution $P$ from one that is merely semidefinite. Furthermore, in Theorem 4.1.1, all checks on existence of Riccati solution are handled within the* lqy *function by standard Riccati solvers (for e.g.,* `dare`,`care` *from* `MATLAB`*).*

The dual of the full-information feedback $H_\infty$ problem is the problem of computing an $H_\infty$ observer gain $H$. This is addressed in the following corollary to Theorem 4.1.1.

## 4.2 $H_\infty$ Full-Control Feedback

**Corollary 4.2.1** ($H_\infty$ Full-control Feedback)**.**
*A solution to Problem 6 exists if and only if the following conditions hold:*

*(i) There exists a full-state feedback* $H^T$ *and matrix* $R_y$ *such that the* $H_\infty$ *CoS identity (3.1.18) holds for the full-information plant* $(G_{FC})^T$ *(2.3.7) and cost matrices* $Q_o, R_o, N_o$.

*(ii) The matrix* $A + H\left(\begin{bmatrix} S_{y21}^T & I_{r_2} \end{bmatrix}\right) C_2$ *is Hurwitz.*

*Furthermore if a solution exists, then for* $X \in RH_\infty$, $\|X\|_\infty < \gamma$, *all solutions are given by*

$$u_2 = \text{lft}(K_{FC}, X) y_2 \tag{4.2.1}$$

*using the "central full-control controller"*

$$K_{FC} = \text{lft}(M_{FC}, S_{FC}), \tag{4.2.2}$$

*where*

$$M_{FC} = \left[\begin{array}{c|c} 0 & H \\ 0 & \begin{bmatrix} -I & 0 \end{bmatrix} \\ \hline I & 0 \end{array}\right], \tag{4.2.3}$$

$$S_{FC} = \left[\begin{array}{c|c} S_{y21}^T & -S_{y_1} \\ I & 0 \\ \hline {S_{y_2}}^{-1} & 0 \end{array}\right]. \tag{4.2.4}$$

*Proof.* See Appendix.

## 4.3 $H_\infty$ Observer

The $H_\infty$ observer for a plant $G$ in (2.3.1) is given by

$$\delta\hat{x} = A\hat{x} + B_2 u_2 + H \begin{bmatrix} \hat{y}_1 \\ \nu \end{bmatrix} \tag{4.3.1}$$

$$\hat{y}_1 = C_1\hat{x} + D_{12}u_2 \tag{4.3.2}$$

$$\hat{y}_2 = C_2\hat{x} + D_{22}u_2 \tag{4.3.3}$$

$$\nu = \hat{y}_2 - y_2 \tag{4.3.4}$$

$$\hat{\tilde{u}}_1 = -S_{y_2}^{-1}\nu \tag{4.3.5}$$

**Lemma 4.3.1** ($H_\infty$ observer ).

*Suppose the feedback gain $H$ exists and the observer equations (4.3.1-4.3.5) for the plant $G$ are used to compute an estimate $\hat{x}$. Now, consider the equations of the squared down plant (3.2.6)*

$$\begin{bmatrix} \delta x \\ \tilde{y}_1 \\ y_2 \end{bmatrix} = S(\tilde{G}) \begin{bmatrix} x \\ \tilde{u}_1 \\ u_2 \end{bmatrix},$$

*then the observer error $e \overset{\Delta}{=} x - \hat{x}$ satisfies*

$$\delta e = (A + HC)e$$

*where $(A + HC)$ is Hurwitz.*

*Proof.* The error dynamics are given by

$$\begin{aligned}
\delta e &= \delta x - \delta \hat{x} \\
&= \tilde{A}x - H_2 S_{y_2} u_1 + \tilde{B}_2 u_2 - (\tilde{A}\hat{x} + \tilde{B}_2 u_2 + H_2 \nu) \\
&= (\tilde{A} + H_2 C_2)e \\
&= (A + H_1 C_1 + H_2 C_2)e \\
&= A_e e \\
&= (A + HC)e
\end{aligned}$$

We note that $(A + HC)$ is Hurwitz since $H^T$ is a stabilizing feedback that solves problem 7 for $G^T$ and cost matrices $Q_o, R_o, N_o$. □

If $x(0) = \hat{x}(0) = 0$ then $e(k) = 0 \forall k$ (Discrete-time) or $e(t) = 0 \forall t$ (Continuous-time). Therefore we may substitute $x(k)$ with $\hat{x}(k)$ or $x(t)$ by $\hat{x}(t)$ without affecting $\|T_{\tilde{y}_1 \tilde{u}_1}\|_\infty$, where $T_{\tilde{y}_1 \tilde{u}_1} \triangleq \mathrm{lft}(\tilde{G}, K)$.

## 4.4 Solution methods for the sub-optimal $H_\infty$ Control Problem

Consider the linear time invariant plant

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}$$

which has descriptor form state-space representation

$$
\begin{aligned}
G(s) \quad &\overset{\text{SS}}{=} \left[\begin{array}{c|cc} -Es+A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \\
&\triangleq \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (Es-A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix}.
\end{aligned}
$$

with state $x \in \mathbb{R}^n$, inputs $u_1 \in \mathbb{R}^{r_1}, u_2 \in \mathbb{R}^{r2}$, and Outputs $y_1 \in \mathbb{R}^{m_1} y_2 \in \mathbb{R}^{m_2}$. When the feedback law $u_2(s) = K(s)y_2(s)$ is applied to the plant $G(s)$, we have

$$
\begin{aligned}
T_{y_1u_1}(s) \quad &= \quad \text{lft}\,(G(s), K(s)) &\quad (4.4.1) \\
&= \quad G_{11}(s) + G_{12}(s)K(s)(I - G_{22}(s)K(s))^{-1}G_{21}(s) &\quad (4.4.2)
\end{aligned}
$$

In this part we make the following assumptions:

**A1**: $(A, B_2, C_2)$ is stabilizable and detectable

**A2**: $D_{12}$ has full column rank and $D_{21}$ has full row rank

**A3**: $\begin{bmatrix} -sE + A & B_2 \\ C_1 & D_{12} \end{bmatrix}$ and $\begin{bmatrix} -sE^T + A^T & {C_2}^T \\ {B_1}^T & {D_{21}}^T \end{bmatrix}$ have no zeros on the j$\omega$ axis and $r_2 \leq m_1$and $m_2 \leq r_1$

**A4**: The plant matrix $E$ is invertible

Note that **A1**,**A2** and **A3** are standard assumptions, required for well-posedness of the all-solutions controller formula of [Limebeer et al., 1988]. The last assumption **A4** might be inessential, but is required here to ensure that the plant is proper and can be transformed to the standard state-space form of [Limebeer et al., 1988, Thm. 5.1'] upon which we base the derivations in this paper.

Eigenspace methods for solving the Optimal $H_\infty$ control problem include (A) the original Hamiltonian matrix methods (e.g., [Glover and Doyle, 1988, Limebeer et al., 1988]), and (B) the more numerically robust extended matrix-pencil methods (e.g., [Benner et al., 2007, K.C.Goh and M.G.Safonov, 1993, Gahinet and Pandey, 1991]). We briefly describe key elements of each of these two methods below.

### 4.4.1 Hamiltonian matrix approach to the sub-optimal $H_\infty$ control problem

[Benner et al., 2007] Let us consider

$$H = \begin{bmatrix} F & -G \\ -K & -F^T \end{bmatrix} \tag{4.4.3}$$

to be a Hamiltonian matrix, where $G, K$ are symmetric and $F, G, K \in R^{n,n}$. $H$ has no eigenvalues on the j$\omega$ axis and the eigenvalues of $H$ have spectral symmetry. We also note that to each Hamiltonian matrix there corresponds an algebraic Riccati equation of the form

$$F^T X + XF + K - XGX = 0. \tag{4.4.4}$$

A solution X of the above equation is said to be *stabilizing* if X=$X^T$ and F-GX is Hurwitz.

Let us define as in [Benner et al., 2007] the symmetric matrices depending on $\gamma$ as a parameter to be:

$$R_H = \begin{bmatrix} {D_{11}}^T \\ {D_{12}}^T \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{r1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.4.5}$$

$$R_J = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix} \begin{bmatrix} {D_{11}}^T & {D_{21}}^T \end{bmatrix} - \begin{bmatrix} \gamma^2 I_{m1} & 0 \\ 0 & 0 \end{bmatrix} \tag{4.4.6}$$

and take $\hat{\gamma}_H$ = max($\gamma \in \mathbb{R} \mid R_H$ is singular), $\hat{\gamma}_J$ = max($\gamma \in \mathbb{R} \mid R_J$ is singular) and $\hat{\gamma}$ = max($\hat{\gamma}_H, \hat{\gamma}_J$)

Finally let us define the Hamiltonian matrices as in [Benner et al., 2007]

$$H(\gamma) = \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} + \begin{bmatrix} -B_1 & -B_2 \\ C_1^T D_{11} & C_1^T D_{12} \end{bmatrix} R_H^{-1} \begin{bmatrix} D_{11}^T C_1 & B_1^T \\ D_{12}^T C_1 & B_2^T \end{bmatrix} \tag{4.4.7}$$

$$J(\gamma) = \begin{bmatrix} A^T & 0 \\ -B_1 B_1^T & -A \end{bmatrix} + \begin{bmatrix} -C_1^T & -C_2^T \\ B_1 D_{11}^T & B_1 D_{21}^T \end{bmatrix} R_J^{-1} \begin{bmatrix} D_{11} B_{11}^T & C_1 \\ D_{21} B_1^T & C_2 \end{bmatrix}. \tag{4.4.8}$$

Under the assumptions A1-A3, for the given plant with $R_H$ and $R_J$ as defined above there exists an internally stabilizing controller such that $\|T_{y_1 u_1}(s)\|_\infty < \gamma$ if and only if the following conditions hold [Benner et al., 2007, Glover and Doyle, 1988, Zhou et al., 1995].

1. $\gamma > \hat{\gamma}$ where $\hat{\gamma}$ is as defined above.

2. There exists a positive semidefinite stabilizing solution $X_H$ for the algebraic Riccati equation associated with H($\gamma$).

3. There exists a positive semidefinite stabilizing solution $X_J$ for the algebraic Riccati equation associated with J($\gamma$).

4. $\gamma^2 > \rho(X_H X_J)$.

### 4.4.2 Matrix pencil approach to the sub-optimal $H_\infty$ control problem

We note that there are numerical difficulties associated with the explicit solution of the Riccati equations and the spectral radius condition. In order to overcome such problems, a matrix-pencil reformulation of the above conditions has been developed [Gahinet and Pandey, 1991, K.C.Goh and M.G.Safonov, 1993, Benner et al., 2007].
In [K.C.Goh and M.G.Safonov, 1993], the following two matrix pencils replace the Hamiltonian matrices corresponding to the two Riccati equations of [Limebeer et al., 1988]:

$$M_{12}(s) = -sM_{E_{12}} + M_{A_{12}} \tag{4.4.9}$$

$$M_{12}(s) = \begin{bmatrix} 0 & -sE + A & B_1 & B_2 \\ sE^T + A^T & {C_1}^T C_1 & {C_1}^T D_{11} & {C_1}^T D_{12} \\ {B_1}^T & {D_{11}}^T C_1 & -\gamma^2 I + {D_{11}}^T D_{11} & {D_{11}}^T D_{12} \\ {B_2}^T & {D_{12}}^T C_1 & {D_{12}}^T D_{11} & {D_{12}}^T D_{12} \end{bmatrix} \tag{4.4.10}$$

$$M_{21}(s) = -sM_{E_{21}} + M_{A_{21}} \tag{4.4.11}$$

$$M_{21}(s) = \begin{bmatrix} 0 & -sE^T + A^T & C_1{}^T & C_2{}^T \\ sE + A & B_1 B_1{}^T & B_1 D_{11}{}^T & B_1 D_{21}{}^T \\ C_1 & D_{11} B_1{}^T & -\gamma^2 I + D_{11} D_{11}{}^T & D_{11} D_{21}{}^T \\ C_2 & D_{21} B_1{}^T & D_{21} D_{11}{}^T & D_{21} D_{21}{}^T \end{bmatrix} \tag{4.4.12}$$

In the extended matrix pencil framework developed by [Benner et al., 2007], $M_{12}(s)$, $M_{21}(s)$ are replaced $\hat{M}_{12}(s)$ and $\hat{M}_{21}(s)$ respectively, where $\hat{M}_{12}(s)$ is defined as

$$\hat{M}_{12}(s) = \begin{bmatrix} 0 & -sE + A & B_1 & B_2 & 0 \\ sE^T + A^T & 0 & 0 & 0 & C_1{}^T \\ B_1{}^T & 0 & -\gamma^2 I & 0 & D_{11}{}^T \\ B_2{}^T & 0 & 0 & 0 & D_{12}{}^T \\ 0 & C_1 & D_{11} & D_{12} & -I \end{bmatrix}, \tag{4.4.13}$$

and $\hat{M}_{21}(s)$ is defined as

$$\hat{M}_{21}(s) = \begin{bmatrix} 0 & -sE^T + A^T & C_1{}^T & C_2{}^T & 0 \\ sE + A & 0 & 0 & 0 & B_1 \\ C_1 & 0 & -\gamma^2 I & 0 & D_{11} \\ C_2 & 0 & 0 & 0 & D_{21} \\ 0 & B_1{}^T & D_{11}{}^T & D_{21}{}^T & -I \end{bmatrix} \tag{4.4.14}$$

$\Xi_{12} = \begin{bmatrix} \Phi_{12} \\ X_{12} \\ V_{12} \\ U_{12} \end{bmatrix}$ and $\Xi_{21} = \begin{bmatrix} \Phi_{21} \\ X_{21} \\ V_{21} \\ U_{21} \end{bmatrix}$ form the bases of the eigenspaces corresponding to $C^-$zeros of $M_{12}(s)$ and $M_{21}(s)$. Similarly the bases for the generalized eigenspaces of $\hat{M}_{12}(s)$ and $\hat{M}_{21}(s)$ can be expressed as

$\hat{\Xi}_{12} = \begin{bmatrix} \Phi_{12} \\ X_{12} \\ V_{12} \\ U_{12} \\ W_{12} \end{bmatrix}$ and $\hat{\Xi}_{21} = \begin{bmatrix} \Phi_{21} \\ X_{21} \\ V_{21} \\ U_{21} \\ W_{21} \end{bmatrix}$

The extended matrix pencils $\hat{M}_{12}(s), \hat{M}_{21}(s)$ of [Benner et al., 2007] have the advantage over the pencils $M_{12}(s), M_{21}(s)$ that they are defined directly in terms of the plant data without the need for potentially data-corrupting multiplication or addition.

Methods for extracting the eigenspaces of matrix pencils has been given in [Dooren, 1981, Dooren, 1979]. A numerically robust computation of these eigenspaces is crucial for computing the optimal $H_\infty$ cost $\gamma_{opt}$ via $\gamma$-iteration technique. Benner et. [Benner et al., 2007] have indeed developed such an algorithm which preserves the structural symmetry of the eigenspaces implicit in the even structure of the pencils $\hat{M}_{12}(s), \hat{M}_{21}(s)$. As a continuation to the effort of solving the $H_\infty$ control problem, we present the controller formulae based on these inverse free pencils in the following section.

# Chapter 5

# Main Result

## 5.1 $LQ$ Feedback $H_\infty$ "All-solutions" Controller Formula

**Theorem 5.1.1** ($H_\infty$ "All-solutions" Controller Formula)**.**
*Given a plant $G$, a solution to the $H_\infty$ control problem exists if and only if the following existence conditions hold:*

*(i) For the plant $G$, there exists a stabilizing solution $H$ to the corresponding $H_\infty$ Full-control problem.*

*(ii) For the squared-down plant $\tilde{G}$, there exists a stabilizing solution $\tilde{F}$ to the corresponding $H_\infty$ Full-information problem.*

*When the above conditions hold, then, for $X \in RH_\infty$, $\|X\|_\infty < \gamma$, the reconstructed-state output-feedback $H_\infty$ "all-solutions" controller is given by*

$$u_2 \;=\; \mathrm{lft}(\tilde{K}_{FI}, X)\begin{bmatrix} \hat{x} \\ \hat{\tilde{u}}_1 \end{bmatrix}, \tag{5.1.1}$$

*where the "central full-information controller"*

$$\tilde{K}_{FI} \;=\; \mathrm{lft}(\tilde{M}_{FI}, \tilde{S}_{FI}), \tag{5.1.2}$$

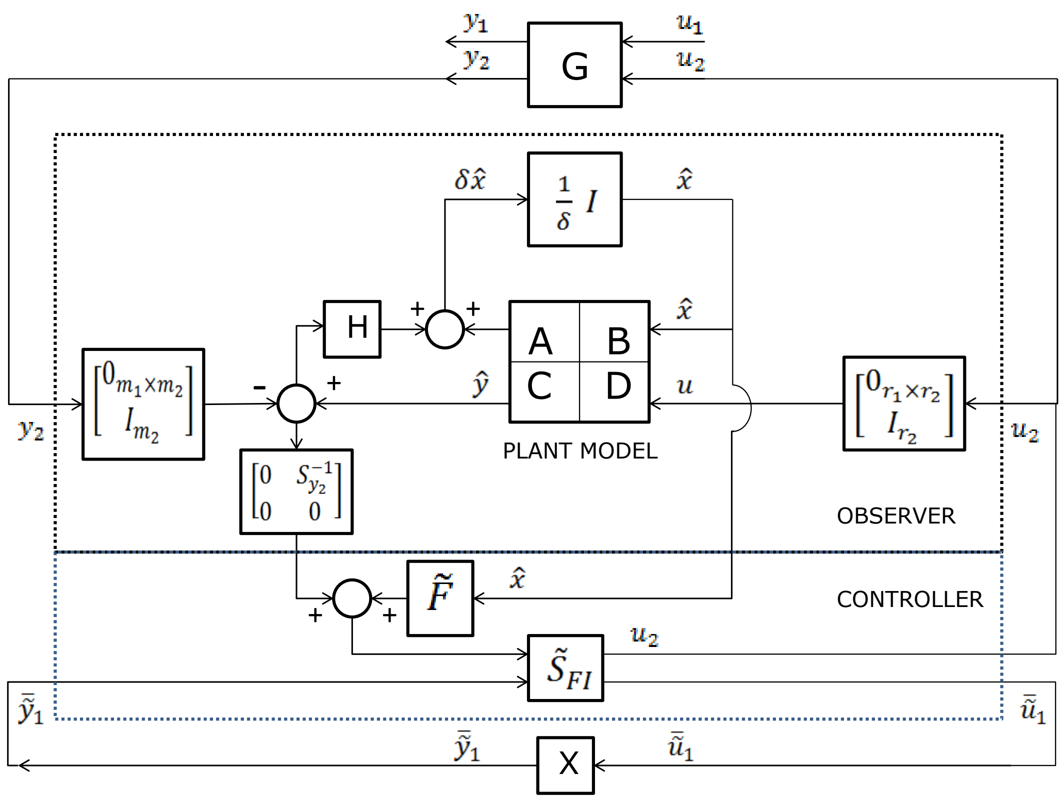


Figure 5.1: A representation of $H_\infty$ "all-solutions" controller

*and $\tilde{M}_{FI}, \tilde{S}_{FI}$ are obtained by applying Theorem 4.1.1 to the squared-down plant $\tilde{G}$. Furthermore, the reconstructed full-information vector comprising of state estimate $\hat{x}$ and exogenous input $\hat{\tilde{u}}_1$ is given by the $H_\infty$ observer (4.3.1-4.3.5).*

*Proof.* See appendix. □

## 5.2 Matrix Pencil $H_\infty$ "All-solutions" Controller Formula

Let us define:

$$\mathbf{\Pi}(\mathrm{s}) = \begin{bmatrix} \gamma^{-2}(sE^T + A^T) & 0 & 0 & 0 & 0 \\ 0 & -sE + A & B_1 & B_2 & 0 \\ 0 & C_1 & D_{11} & D_{12} & 0 \\ 0 & C_2 & D_{21} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & -\gamma^{-2}{D_{11}}^T \end{bmatrix} \tag{5.2.1}$$

also define:[K.C.Goh and M.G.Safonov, 1993]

$$\hat{D}_{11} = {D_{11}}^T(\gamma^2 I - D_{11}{D_{11}}^T)^{-1} \tag{5.2.2}$$

$$\hat{D}_{12} = [{D_{12}}^T(I - \gamma^{-2}D_{11}{D_{11}}^T)^{-1}D_{12}]^{\frac{1}{2}} \tag{5.2.3}$$

$$\hat{D}_{21} = [D_{21}(I - \gamma^{-2}{D_{11}}^T D_{11})^{-1}{D_{21}}^T]^{\frac{1}{2}} \tag{5.2.4}$$

Q(s) is defined as any $r_2 \times m_2$ stable transfer function matrix such that ([K.C.Goh and M.G.Safonov, 1993])

$$\|\hat{D}_{12}Q(s)\hat{D}_{21}\|_\infty < \gamma$$

**Theorem 5.2.1** (Matrix Pencil $H_\infty$ Controller Formulae)**.**
*Suppose $\bar{\sigma}(D_{11}) < \gamma$. Then, the sub-optimal $H_\infty$ control problem for the given plant has an internally stabilizing controller $K_Q(s)$, provided the conditions 1 to 4 hold.*
*The internally stabilizing controller is then given by:*

$$K_Q(s) = \mathbf{F}(K(s), Q(s))$$

*The descriptor representation of K(s) is as follows:*

$$K(s) \overset{des}{=} \left[\begin{array}{c|cc} -sE_k + A_k & B_{k1} & B_{k2} \\ \hline C_{k1} & D_{k11} & D_{k12} \\ C_{k2} & D_{k21} & D_{k22} \end{array}\right] \tag{5.2.5}$$

*Where:*

$$[-sE_k + A_k] = [\hat{\Xi}_{21}^T \Pi(s) \hat{\Xi}_{12}] \tag{5.2.6}$$

$$\begin{bmatrix} B_{k1} & B_{k2} \end{bmatrix} = \begin{bmatrix} \hat{\Xi}_{21}^T \end{bmatrix} \left[ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -I_{m2} \\ 0 \end{pmatrix} \quad \Pi(s) \begin{pmatrix} 0 \\ 0 \\ 0 \\ I_{r2} \\ 0 \end{pmatrix} \right] \tag{5.2.7}$$

$$\begin{bmatrix} C_{k1} \\ C_{k2} \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 0 & 0 & 0 & I_{r2} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 & -I_{m2} & 0 \end{pmatrix} \Pi(s) \end{bmatrix} \begin{bmatrix} \hat{\Xi}_{12} \end{bmatrix} \tag{5.2.8}$$

$$\begin{bmatrix} D_{k11} & D_{k12} \\ D_{k21} & D_{k22} \end{bmatrix} = \begin{bmatrix} 0 & I_{r2} \\ I_{m2} & -D_{22} + D_{21}\hat{D}_{11}D_{12} \end{bmatrix} \tag{5.2.9}$$

*Proof.* See appendix. □

# Chapter 6

# Examples

**Example 1.** *Given the plant*

$$P(s) = \frac{(s-1)}{(s+1)^2}, \tag{6.0.1}$$

*consider the "mixed" sensitivity $H_\infty$ control problem (Problem 2) for*

$$T_{y_1u_1} \triangleq \begin{bmatrix} W_p(s)1/(1+P(s)K(s)) \\ W_u(s)K/(1+P(s)K(s)) \\ W_t(s)P(s)K(s)/(1+P(s)K(s)) \end{bmatrix}, \tag{6.0.2}$$

*with the following choice of "weights"*

$$W_p(s) = \frac{0.1(s+100)}{(100s+1)}, \tag{6.0.3}$$

$$W_u(s) = 0.1, \tag{6.0.4}$$

$$W_t(s) = 0. \tag{6.0.5}$$

*For $T_{y_1u_1}$ in equation (6.0.2) the generalized plant $G$ is given by (see Sec.3.8.1, [Skogestad and Postlethwaite, 2005], also see Fig.6.1)*

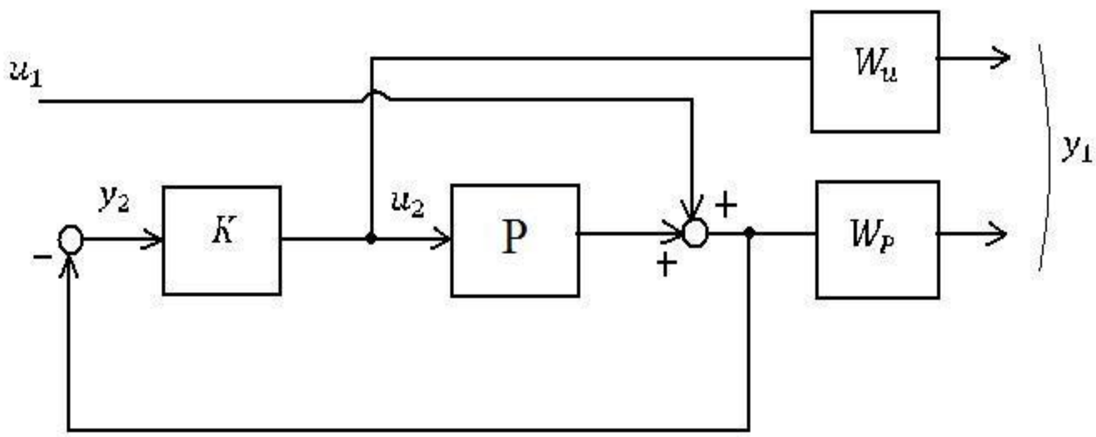


Figure 6.1: Mixed Sensitivity $H_\infty$ control problem

$$G(s) \quad = \left[\begin{array}{c|c} W_p(s) & -W_p(s)P(s) \\ 0 & W_u(s) \\ 0 & W_t(s)P(s) \\ \hline 1 & -P(s) \end{array}\right]. \tag{6.0.6}$$

*An $H_\infty$ optimal controller is then computed using our main Theorem 5.1.1 (see summary in Appendix:Table 6.1). From Fig. 6.2 we see that all design requirements were met as our result agrees with the output of* `MATLAB`*'s routine* `mixsyn` *(see [Matlab, 2010b]).*

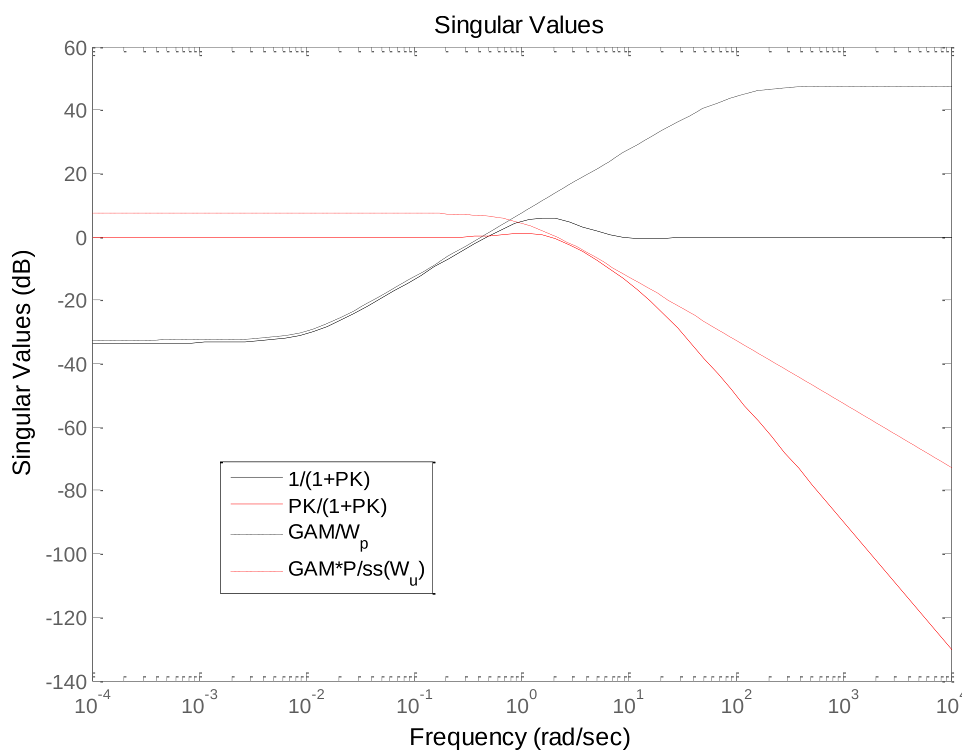


Figure 6.2: Singular-value Bode plot of closed-loop functions

Table 6.1: $H_\infty$ Controller Design Summary

| Design Parameter Values (refer to Fig.5.1) | |
|---|---|
| $A$ | $\begin{bmatrix} -0.01000 & 0.22096 & -0.15624 \\ 0 & -1 & 1.41421 \\ 0 & 0 & \end{bmatrix}$ |
| $B$ | $\begin{bmatrix} 0.31248 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}$ |
| $C$ | $\begin{bmatrix} 0.31998 & 0.00071 & -0.00050 \\ 0 & 0 & 0 \\ 0 & 0.70711 & -0.50000 \end{bmatrix}$ |
| $D$ | $\begin{bmatrix} 0.00100 & 0 \\ 0 & 0.10000 \\ 1 & 0 \end{bmatrix}$ |
| $H$ | $\begin{bmatrix} 0 & 0 & -0.31248 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ |
| $S_{y_2}$ | [1] |
| $\tilde{F}$ | $\begin{bmatrix} 43.66786 & 8.14796 & 2.89797 \\ -102.75678 & -19.42774 & -7.12868 \end{bmatrix}$ |
| $\tilde{S}_{FI}$ | $\begin{bmatrix} 0 & 1 & 10 \\ -0.99999 & 0 & 0 \end{bmatrix}$ |
| $X$ | [0] |

# Chapter 7

# Conclusion

Building upon the seminal works of [Glover and Doyle, 1989], [Doyle et al., 1989], [Limebeer et al., 1988] in the continuous-time case and prominent results such as [Limebeer et al., 1989, Green and Limebeer, 1995, Petkov et al., 1999, Iglesias and Glover, 1991, Ionescu et al., 1999, Stoorvogel et al., 1994] in the discrete-time case, we present a unified formula (5.1.1) and representation structure (Fig.5.1) for $H_\infty$ "all-solutions" controllers. With our focus on input-output weighting "cost" functions, we revisit the "completion of squares" identity (Lemma 3.1.1) to show that the continuous and discrete-time cases differ only in the choice of a "completion of squares" matrix $R_u$ (3.1.2). With a simpler set of existence conditions we see that this result can be easily implemented in software to handle continuous and discrete-time plants alike, without the hassle of bilinear transforms and "loop-shifting" transformations. This result is simpler than any earlier formula appearing in the aforementioned works (e.g., [Doyle et al., 1989, Iglesias and Glover, 1991] etc.). Otherwise intricate pencils and/or Hamiltonians are eliminated from both our derivations and controller formula. Instead these details become inessential details in subroutines of established $LQ$ solution formulae. The controller realization preserves and extends to the general case the internal plant model controller structure first identified by [Glover and Doyle, 1989]. As shown in Figure 5.1, the general "all-solutions" $H_\infty$ controller realization derived in this thesis contains at its core an $H_\infty$ optimal state-estimator with an exact copy of the plant model.

The second main result of this thesis builds upon the work of [Benner et al., 2007] which gives us a numerically robust even matrix pencil algorithm for computing the optimal value of $\gamma$ via $\gamma$-iteration, we have followed up in this work with simplified matrix pencil formulae for the all-solutions $H_\infty$ controller too. A significant feature of our formulae is that each element of the pencils is expressed directly in terms of the original descriptor-form state space matrices of the plant and the even pencil eigenspaces computed by the even pencil algorithm of [Benner et al., 2007], so that there are no data-corrupting numerical operations required to form any of the matrices that appear in our "all-solutions" controller formulae.

# Bibliography


[Benner et al., 2007] Benner, P., Byers, R., and Xu, H. (2007). A robust numerical method for the $\gamma$-iteration in $H_\infty$ control. *Linear Algebra and Applications*, 425(2-3):548–570.

[Brockett, 1970] Brockett, R. (1970). *Finite Dimensional Linear Systems*. Wiley, NY.

[Dooren, 1979] Dooren, P. (1979). The computation of kronecker's canonical form of a singular pencil. *Linear Algebra and its Applications*, 27:103–140.

[Dooren, 1981] Dooren, P. V. (1981). A generalized eigenvalue approach for solving riccati equations. *SIAM Journal of Scientific and Statistical Computing*, 2(2):262–283.

[Doyle et al., 1989] Doyle, J., Glover, K., Khargonekar, P., and Francis, B. (1989). State space solutions to standard $H_2$ and $H_\infty$ control problems. *IEEE Transactions in Automatic Control*, AC-34(8):731–747.

[Gahinet and Pandey, 1991] Gahinet, P. M. and Pandey, P. (1991). Fast and numerically robust algorithm for computing the $H_\infty$ optimum. In *Proceedings of 30th IEEE Conference on Decision and Control*, Brighton, England.

[Glover and Doyle, 1988] Glover, K. and Doyle, J. (1988). State space formulae for all stabilizing controllers that satisfy an $H_\infty$ norm bound and relations to risk sensitivity. *Systems and Control Letters*, 11:167–172.

[Glover and Doyle, 1989] Glover, K. and Doyle, J. C. (1989). A state-space approach to $H_\infty$ optimal control. In *Three Decades of Mathematical System Theory*, pages 179–218. Springer-Verlag, NY.

[Green and Limebeer, 1995] Green, M. and Limebeer, D. (1995). *Linear Robust Control*. Prentice Hall.

[Iglesias and Glover, 1991] Iglesias, P. and Glover, K. (1991). State-space approach to discrete-time $H_\infty$ control. *International Journal of Control*, 54(5):1031–73.

[Ionescu et al., 1999] Ionescu, V., Oara, C., and Weiss, M. (1999). *Generalized Riccati theory and robust control: A Popov function approach*. John Wiley.

[Karthikeyan and Safonov, 2009] Karthikeyan, A. and Safonov, M. G. (2009). $LQ$ feedback formulation for discrete-time $H_\infty$ output feedback. In *Proceedings of IEEE Conference on Decision and Control*, Shanghai, CN.

[Karthikeyan and Safonov, 2010] Karthikeyan, A. and Safonov, M. G. (2010). $LQ$ approach to an all-solutions formula for $H_\infty$ output feedback control. In *Proceedings of IEEE Conference on Decision and Control*, Atlanta, GA.

[Karthikeyan and Safonov, 2012] Karthikeyan, A. and Safonov, M. G. (2012). A simple unified formula for discrete and continuous-time $H_\infty$ "all-solutions" controllers. In *Proceedings of* $7^{th}$ *IFAC Symposium on Robust Control Design*, Aalborg, Denmark.

[K.C.Goh and M.G.Safonov, 1993] K.C.Goh and M.G.Safonov (1993). $H_\infty$ control:inverse free formulation for $D_{11} \neq 0$ and eliminating pole-zero cancellations via interpolation. In *Proceedings of 32nd IEEE Conference on Decision and Control*, San Antonio, TX.

[Lancaster and Rodman, 1995] Lancaster, P. and Rodman, L. (1995). *Algebraic Riccati Equations*. Oxford Science Publications.

[Limebeer et al., 1989] Limebeer, D., Green, M., and Walker, D. (1989). Discrete time $H_\infty$ control. In *Proceedings of Conference on Decision and Control*, pages 392–396, Tampa, FL.

[Limebeer et al., 1988] Limebeer, D. J., Kasenally, E. M., Jaimouka, I., and Safonov, M. G. (1988). All solutions to the four block general distance problem. In *Proceedings of IEEE Conference on Decision and Control*, Austin, TX.

[Mageirou and Ho, 1977] Mageirou, E. and Ho, Y. (1977). Decentralized stabilization via game theoretic methods. *Automatica*, 13:393–399.

[Matlab, 2010a] Matlab (2001-2010a). `lft` - Generalized feedback interconnection of two LTI models. In *Control System Toolbox*. MathWorks, Natick, MA.

[Matlab, 2010b] Matlab (2001-2010b). `mixsyn` - $H_\infty$ mixed-sensitivity synthesis method for robust control. In *Robust Control Toolbox*. MathWorks, Natick, MA.

[Matlab, 2008] Matlab (2008). *Control Systems Toolbox User's Guide*. MathWorks, Natick, MA.

[Medanic, 1967] Medanic, J. (1967). Bounds on the performance index and the Riccati equation in differential games. *IEEE Transactions in Automatic Control*, AC-7(5):613–614.

[Petkov et al., 1999] Petkov, P., Gu, D., and Konstantinov, M. (1999). Fortran 77 routines for $H_\infty$ and $H_2$ design of discrete-time linear control systems. In *NICONET*. WGS.

[Redheffer, 1960] Redheffer, A. (1960). On a certain linear fractional transformation. *Journal of Math. Physics*, 39(7):269–286.

[Safonov et al., 1989] Safonov, M. G., Limebeer, D. J., and Chiang, R. Y. (1989). Simplifying the $H_\infty$ theory via loop-shifting, matrix-pencil and descriptor concepts. *International Journal of Control*, 50(6):2467–2488.

[Skogestad and Postlethwaite, 2005] Skogestad, S. and Postlethwaite, I. (2005). *Multivariable Feedback Control, 2nd Edition*. Wiley, NY.

[Stoorvogel et al., 1994] Stoorvogel, A., Saberi, A., and Chen, B. (1994). The discrete-time $H_\infty$ control problem with measurement feedback. *International Journal of Robust and Nonlinear Control*, 4:457–479.

[Zhou et al., 1995] Zhou, K., Doyle, J., and K.Glover (1995). *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ.

# Appendix A

# Proofs

## A.1 Proofs for $LQ$ Feedback formulation theorems for $H_\infty$ Full-Information, $H_\infty$ Full-Control and $H_\infty$ Output Feedback "All-solutions" cases

*Theorem 4.1.1:$H_\infty$ Full-Information Feedback.* [Karthikeyan and Safonov, 2010, Karthikeyan and Safonov, 2012, Theorem 1,Theorem 11]

By Lemma 3.2.1, a controller $K_{FI}$ solves the standard $H_\infty$ problem for the full information plant $S(G_{FI})$ if (a) it stabilizes $G_{FI}$ and (b) it solves the $H_\infty$ control problem for the corresponding squared down plant

$$S(\bar{G}_{FI}) = \left[\begin{array}{c|cc} \bar{A} & B_1 S_{u_1}^{-1} & B_2 \\ \hline -S_{u_2}F_2 & S_{u_2}S_{u_{21}}S_{u_1}^{-1} & S_{u_2} \\ \begin{bmatrix} I \\ F_1 \end{bmatrix} & \begin{bmatrix} 0 \\ S_{u_1}^{-1} \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]$$

where $\bar{A} = A + B_1 F_1$.

Denote by $X$ the closed-loop system

$$X \triangleq \mathrm{lft}(\bar{G}_{FI}, K) \tag{A.1.1}$$

so that

$$\bar{y}_1 = X\bar{u}_1 \ \forall \bar{u}_1. \tag{A.1.2}$$

From the first output equation of $S(\bar{G}_{FI})$, we have

$$\bar{y}_1 = -S_{u_2}F_2x + S_{u_2}S_{u_{21}}S_{u_1}^{-1}\bar{u}_1 + S_{u_2}u_2 \ \forall \bar{u}_1, u_2 \tag{A.1.3}$$

Substituting equation (A.1.2) into (A.1.3) we have

$$X\bar{u}_1 = -S_{u_2}F_2x + S_{u_2}S_{u_{21}}S_{u_1}^{-1}\bar{u}_1 + S_{u_2}u_2 \ \forall \bar{u}_1, u_2$$

solving for $u_2$ in terms of $x, \bar{u}_1$, we obtain the $H_\infty$ full-information control law (4.1.1).

$$\begin{aligned} u_2 &= u_2^* - (S_{u_{21}} - {S_{u_2}}^{-1}XS_{u_1})(u_1 - u_1^*) & \text{(A.1.4)} \\ &= \text{lft}(K_{FI}, X)\begin{bmatrix} x \\ u_1 \end{bmatrix} & \text{(A.1.5)} \end{aligned}$$

So, from Lemma 3.2.1 the result follows provided $K$ stabilizes $G_{FI}$.

The system $T_{y_1u_1}$ can be decomposed as

$$\begin{aligned} T_{y_1u_1} &= \text{lft}(G, \text{lft}(K_{FI}, X)) & \text{(A.1.6)} \\ &= \text{lft}(T, X) & \text{(A.1.7)} \end{aligned}$$

where $T \triangleq \text{lft}(G, K_{FI})$. Now, by (3.1.19) we have

$$u_2 = u_2^* - S_{u_{21}}(u_1 - u_1^*). \tag{A.1.8}$$

Substituting (A.1.8) and (3.1.19) into (2.3.1), we find that for all $u_1, \bar{y}_1$ it holds that

$$\begin{bmatrix} y_1 \\ \bar{u}_1 \end{bmatrix} = T \begin{bmatrix} u_1 \\ \bar{y}_1 \end{bmatrix} \tag{A.1.9}$$

where $T$ has state-space representation

$$S(T) = \left[\begin{array}{c|cc} A + B_2(F_2 + S_{u_{21}} F_1) & B_1 - B_2 S_{u_{21}} & B_2 S_{u_2}^{-1} \\ \hline C_1 + D_{12}(F_2 + S_{u_{21}} F_1) & (I - S_{u_{21}}) D_{11} & D_{12} S_{u_2}^{-1} \\ -S_{u_1} F_1 & S_{u_1} & 0 \end{array}\right] \tag{A.1.10}$$

for all $x, u_2, y_1, y_2, \bar{u}_1, \bar{u}_2$ satisfying the system equations (16)-(22) and (26). By condition iii) of Theorem 4.1.1, $T$ is stable and hence we have $T_{y_1 u_1} = \mathrm{lft}(T, X)$ is stable for the special case $X = 0$. Further, from equation (3.1.18) the completion of the squares Lemma 3.2.1, it holds for all

$$\begin{bmatrix} y_1 \\ \bar{u}_1 \end{bmatrix} = T \begin{bmatrix} u_1 \\ \bar{y}_1 \end{bmatrix}$$

that

$$\left\| \begin{bmatrix} y_1 \\ \gamma \bar{u}_1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \gamma u_1 \\ \bar{y}_1 \end{bmatrix} \right\|. \tag{A.1.11}$$

It follows that $\left\| \begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} T \begin{bmatrix} 1/\gamma & 0 \\ 0 & 1 \end{bmatrix} \right\|_\infty \leq 1$; from which it follows by the small gain stability theorem that $T_{y_1 u_1} \triangleq \mathrm{lft}(T, X)$ is internally stable for all $X \in RH_\infty$ satisfying $\|X\|_\infty < \gamma$. □ □

*Corollary 4.2.1:$H_\infty$ Full-control feedback.*

Applying Theorem 4.1.1 to the transpose of $T_{y_1u_1}$ and transposing back, the result follows immediately. □

The matrix $\begin{bmatrix} H_1 & H_2 \end{bmatrix}$ in Corollary 4.2.1 is called an $H_\infty$ observer gain matrix. Consider the $H_\infty$ observer squared down plant $\tilde{G}$ (3.2.6). Let $\tilde{G}$ have a stabilizing feedback $\tilde{F}$ that solves problem 7 for $Q_c, \tilde{N}_c = 0$ and

$$\tilde{R}_c = \begin{bmatrix} -\gamma^2 I_{m_2 \times m_2} & 0 \\ 0 & 0 \end{bmatrix} \in R^{(m_2+r_2)\times(m_2+r_2)}. \quad \text{(A.1.12)}$$

Also, let $R_{\tilde{u}}$ have a Schur $J$-factorization (2.2.3) with $S_{\tilde{u}_1}, S_{\tilde{u}_2}, S_{\tilde{u}_{21}}$ given by equations (3.1.22-2.2.7). Now, recall from "completion of squares" that the all-solutions controller for the squared-down full-information plant is given by equation (4.1.1) where $X$ is any transfer function such that, $\|X\|_\infty < \gamma$. As the $H_\infty$ observer (4.3.1-4.3.5) has $e(k) = 0 \forall k$ when $x(0) = \hat{x}(0) = 0$, we can replace $x$ with $\hat{x}$ and $\tilde{u}_1$ with $\hat{\tilde{u}}_1$. Thus, we have the proof of our main result 5.1.1.

*Theorem 5.1.1:$H_\infty$ "All-solutions" Controller Formula.*

From Corollary 4.2.1, it follows that the necessary existence conditions for a solution to Problem 1 are the existence of a stabilizing feedback $H$ and $A + (H_2 + H_1 S_{y_{21}}^T)C_2$ being Hurwitz. When these conditions hold, it follows from Lemma 3.2.2 that $K$ solves Problem 1 for the plant $G$ if and only if it solves the problem for the squared-down plant $\tilde{G}$. The results then follow directly from Lemma 4.3.1, and Theorem 4.1.1. □

## A.2 Proof for Matrix pencil "All-solutions" $H_\infty$ Controller formula

We begin by assuming without loss of generality that $E = I$ and $D_{11} = 0$. For $E \neq I$ it suffices to notice that our matrix pencil formulae in Theorem 5.2.1 remain invariant under the change of variables $[A, B_1, B_2, \Phi_{12}, X_{21}] \rightarrow [E^{-1}A, E^{-1}B_1, E^{-1}B_2, E^T\Phi_{12}, E^TX_{21}]$. [This change of variables corresponds to pre-multiplying the first row and post-multiplying the first column of the pencil $\hat{M}_{12}$ by $E^{-1}$ and $(E^{-1})T$ respectively, and similarly pre-multiplying the second row and post-multiplying the second column of the pencil $\hat{M}_{21}(s)$ by $E^{-1}$ and $(E^{-1})^T$ respectively]. For the case $D_{11} \neq 0$, the result follows from [Safonov et al., 1989, Lemma 1], which establishes that $K(s)$ solves the sub-optimal $\text{H}_\infty$ problem for $G(s)$ if and only if it solves the problem for the plant

$$\hat{G}(s) \overset{\text{dss}}{=} \left[\begin{array}{c|cc} -Es + A & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \hat{D_{12}} \\ \hat{C}_2 & \hat{D_{21}} & D_{22} + D_{21}\hat{D_{11}}D_{12} \end{array}\right]$$

which has a zero $D_{11}$-matrix.

Consider the formulae given under Theorem $5.1'$ and Theorem $5.2'$ of [Limebeer et al., 1988] which hold under the assumptions A1 to A3 along with $D_{22} = 0$ and $\|D_{11}\|_2 < \gamma$. Additionally $D_{21}D_{21}^T = I_{m_2}$and ${D_{12}}^T D_{12} = I_{r_2}$.

According to [Limebeer et al., 1988, Theorem 5.2'] all internally stabilizing controllers satisfying $\|F_l(P, K)\|_\infty \leq \gamma$ are given by

$$K(s) \overset{\text{dss}}{=} \left[\begin{array}{c|cc} -sE_k + A_k & B_{k1} & B_{k2} \\ \hline C_{k1} & D_{k11} & D_{k12} \\ C_{k2} & D_{k21} & D_{k22} \end{array}\right] \tag{A.2.1}$$

Where:

$$\begin{aligned}
E_k &= {Y_{\infty 1}}^T X_{\infty 1} - \gamma^{-2} {Y_{\infty 2}}^T X_{\infty 2} \\
B_{k1} &= (\gamma^2 {Y_{\infty 1}}^T B_1 + {Y_{\infty 2}}^T {C_1}^T D_{11} + {Y_{\infty 2}}^T {C_2}^T D_{21}(\gamma^2 I - {D_{11}}^T D_{11})) \\
&\quad \times(\gamma^2 I - \tilde{D}_\perp^T \tilde{D}_\perp {D_{11}}^T D_{11})^{-1} {D_{21}}^T \\
C_{k1} &= -{D_{12}}^T (\gamma^2 I - D_{11} {D_{11}}^T D_\perp {D_\perp}^T)^{-1} \\
&\quad \times(\gamma^2 C_1 X_{\infty 1} + D_{11} {B_1}^T X_{\infty 2} + (\gamma^2 I - D_{11} {D_{11}}^T) D_{12} {B_2}^T X_{\infty 2}) \\
D_{k_{12}} &= (I - {D_{12}}^T D_{11} (\gamma^2 I - {D_{11}}^T D_\perp {D_\perp}^T D_{11})^{-1} {D_{11}}^T D_{12})^{1/2} \\
D_{k_{21}} &= (I - D_{21} {D_{11}}^T (\gamma^2 I - D_{11} \tilde{D}_\perp^T \tilde{D}_\perp D_{11})^{-1} D_{11} {D_{21}}^T)^{1/2} \\
D_{k_{22}} &= -({D_{k21}}^{-1})^T D_{21} {D_{11}}^T (\gamma^2 I - D_{11} \tilde{D}_\perp^T \tilde{D}_\perp {D_{11}}^T)^{-1} D_{12} D_{k12} \\
A_k &= E_k T_x + B_{k1} {D_{k21}}^{-1} C_{k2} \\
B_{k2} &= ({Y_{\infty 1}}^T B_2 + ({Y_{\infty 1}}^T B_1 \tilde{D}_\perp^T \tilde{D}_\perp D_{11}^T + {Y_{\infty 2}}^T ({C_1}^T - {C_2}^T D_{21} {D_{11}}^T)) \\
&\quad \times(\gamma^2 I - \tilde{D}_\perp^T \tilde{D}_\perp {D_{11}}^T D_{11})^{-1} D_{12}) D_{k12} \\
C_{k2} &= -D_{k21} (C_2 X_{\infty 1} + D_{21} (\gamma^2 I - {D_{11}}^T D_\perp {D_\perp}^T D_{11})^{-1} \\
&\quad \times({D_{11}}^T D_\perp {D_\perp}^T C_1 X_{\infty 1} + {B_1}^T X_{\infty 2} - {D_{11}}^T D_{12} {B_2}^T X_{\infty 2}))
\end{aligned}$$

**Remark 6.** *It is known that [Benner et al., 2007, K.C.Goh and M.G.Safonov, 1993, Gahinet and Pandey, 1991] that*

$$\begin{bmatrix} X_{\infty 1}, & X_{\infty 2} \end{bmatrix} = \begin{bmatrix} X_{12}, & \phi_{12} \end{bmatrix}, \tag{A.2.2}$$

$$\begin{bmatrix} Y_{\infty 1}, & Y_{\infty 2} \end{bmatrix} = \begin{bmatrix} X_{21}, & \phi_{21} \end{bmatrix}. \tag{A.2.3}$$

Using this and the fact the $\Xi_{12}, \Xi_{21}$ span the respective right eigenspaces of the pencils $M_{12}(s), M_{21}(s)$, we may simplify each of the terms of the controller and arrive at our modified formulae. Consider first the $B_{k1}$ term.

$$\begin{aligned}
B_{k1} &= (\gamma^2 {Y_{\infty 1}}^T B_1 + {Y_{\infty 2}}^T {C_1}^T D_{11} + {Y_{\infty 2}}^T {C_2}^T D_{21}(\gamma^2 I - {D_{11}}^T D_{11})) \\
&\quad \times (\gamma^2 I - \tilde{D}_\perp^T \tilde{D}_\perp {D_{11}}^T D_{11})^{-1} {D_{21}}^T \\
&= (\gamma^2 {X_{21}}^T B_1 + {\Phi_{21}}^T {C_1}^T D_{11} + {\Phi_{21}}^T {C_2}^T D_{21}(\gamma^2 I - {D_{11}}^T D_{11})) \\
&\quad \times (\gamma^2 I - \tilde{D}_\perp^T \tilde{D}_\perp {D_{11}}^T D_{11})^{-1} {D_{21}}^T \\
&= (\gamma^2 {X_{21}}^T \hat{B}_1 + {\Phi_{21}}^T \hat{C}_2^T \hat{D}_{21} \gamma^2) \gamma^{-2} \hat{D}_{21}^T \\
&= X_{21}^T \hat{B}_1 \hat{D}_{21}^T + {\Phi_{21}}^T \hat{C}_2^T \\
&= X_{21}^T \hat{B}_1 \hat{D}_{21}^T - (\hat{D}_{21} \hat{B}_1^T X_{21} + \hat{D}_{21} \hat{D}_{21}^T U_{21})^T \\
&= X_{21}^T \hat{B}_1 \hat{D}_{21}^T - {X_{21}}^T \hat{B}_1 \hat{D}_{21}^T - {U_{21}}^T \hat{D}_{21} \hat{D}_{21}^T \\
&= -{U_{21}}^T \\
&= -\hat{U}_{21}^T \\
&= \hat{\Xi}_{21}^T \begin{bmatrix} 0 & 0 & 0 & -I_{m2} & 0 \end{bmatrix}^T
\end{aligned}$$

The results for $B_{k2}, C_{k1}, C_{k2}$ and $D_k$ can be derived in a similar fashion.

It remains now to establish the equivalence of our formula for $-sE_k + A_k$ with that of [Limebeer et al., 1988].

Clearly,

$$
\begin{aligned}
E_k &= {Y_{\infty 1}}^T X_{\infty 1} - \gamma^{-2} {Y_{\infty 2}}^T X_{\infty 2} \\
&= {X_{21}}^T X_{12} - \gamma^{-2} {\Phi_{21}}^T \Phi_{12} \\
&= \begin{bmatrix} X_{21} \\ \Phi_{21} \end{bmatrix}^T \begin{bmatrix} I_n & 0 \\ 0 & -\gamma^{-2} I_n \end{bmatrix} \begin{bmatrix} X_{12} \\ \Phi_{12} \end{bmatrix}
\end{aligned}
$$

And,

$$
A_k = \begin{bmatrix} X_{21} \\ \Phi_{21} \end{bmatrix}^T \begin{bmatrix} I_n & 0 \\ 0 & -\gamma^{-2} I_n \end{bmatrix} \begin{bmatrix} X_{12} \\ \Phi_{12} \end{bmatrix} T_x + B_{k1} {D_{k21}}^{-1} C_{k2}
$$

From [Limebeer et al., 1988, Theorem 5.1], we have

$$
\begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} T_x = H_\infty \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix}.
$$

Therefore,

$$
A_k = \begin{bmatrix} X_{21} \\ \Phi_{21} \end{bmatrix}^T \begin{bmatrix} I_n & 0 \\ 0 & -\gamma^{-2} I_n \end{bmatrix} H_\infty \begin{bmatrix} X_{12} \\ \Phi_{12} \end{bmatrix} + B_{k1} {D_{k21}}^{-1} C_{k2}
$$

loop shift and set $\hat{D}_{11} = 0$. The Hamiltonian matrix can then be simplified as follows:

$$
\begin{aligned}
H_\infty &= \begin{bmatrix} H_{\infty_{11}} & H_{\infty_{12}} \\ H_{\infty_{21}} & H_{\infty_{22}} \end{bmatrix} \\
&= \begin{bmatrix} \hat{A} - \hat{B}_2 \hat{D}_{12}^T \hat{C}_1 & \gamma^{-2} \hat{B}_1 \hat{B}_1^T - \hat{B}_2 \hat{B}_2^T \\ -\hat{C}_1^T (I - \hat{D}_{12} \hat{D}_{12}^T) \hat{C}_1 & -\hat{A}^T + \hat{C}_1^T \hat{D}_{12} \hat{B}_2^T \end{bmatrix}
\end{aligned}
$$

Therefore,

$$
-sE_k + A_k = \begin{bmatrix} X_{21} \\ \Phi_{21} \end{bmatrix}^T \begin{bmatrix} -sI + H_{\infty_{11}} & H_{\infty_{12}} \\ -\gamma^{-2} H_{\infty_{21}} & \gamma^{-2} sI - \gamma^{-2} H_{\infty_{22}} \end{bmatrix} \begin{bmatrix} X_{12} \\ \Phi_{12} \end{bmatrix} + B_{k1} {D_{k21}}^{-1} C_{k2}
$$

Simplifying using the pencils $M_{12}$ and $M_{21}$ with $\hat{D}_{11} = 0$

$$
\begin{aligned}
-sE_k + A_k &= \begin{bmatrix} {\Phi_{21}}^T & {X_{21}}^T & {V_{21}}^T & {U_{21}}^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} \gamma^{-2}[sI + A^T - {\hat{C}_1}^T \hat{D}_{12}\hat{B}_2^T] & \gamma^{-2}\hat{C}_1^T(I - \hat{D}_{12}\hat{D}_{12}^T)\hat{C}_1 & 0 & 0 \\ \gamma^{-2}\hat{B}_1\hat{B}_1^T - \hat{B}_2\hat{B}_2^T & -sI + \hat{A} - \hat{B}_2\hat{D}_{12}\hat{C}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} \\
&\quad \times \begin{bmatrix} {\Phi_{12}}^T & {X_{12}}^T & {V_{12}}^T & {U_{12}}^T \end{bmatrix}^T \\
&= {\Xi_{21}}^T \begin{bmatrix} \gamma^{-2}[sI + \hat{A}^T - \hat{C}_1^T \hat{D}_{12}\hat{B}_2^T] & \gamma^{-2}\hat{C}_1^T(I - \hat{D}_{12}\hat{D}_{12}^T)\hat{C}_1 & 0 & 0 \\ -\hat{B}_2\hat{B}_2^T & -sI + \hat{A} - \hat{B}_2\hat{D}_{12}\hat{C}_1 & \hat{B}_1 & 0 \\ 0 & \hat{C}_1 & 0 & 0 \\ 0 & \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} \Xi_{12} \\
&= {\Xi_{21}}^T \begin{bmatrix} \gamma^{-2}[sI + \hat{A}^T - \hat{C}_1^T \hat{D}_{12}\hat{B}_2^T] & \gamma^{-2}\hat{C}_1^T(I - \hat{D}_{12}\hat{D}_{12}^T)\hat{C}_1 & 0 & 0 \\ 0 & -sI + \hat{A} & \hat{B}_1 & \hat{B}_2 \\ 0 & \hat{C}_1 & 0 & 0 \\ 0 & \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} \Xi_{12} \\
&= {\Xi_{21}}^T \begin{bmatrix} \gamma^{-2}[sI + \hat{A}^T - \hat{C}_1^T \hat{D}_{12}\hat{B}_2^T] & 0 & 0 & 0 \\ 0 & -sI + \hat{A} & \hat{B}_1 & \hat{B}_2 \\ 0 & (I - \hat{D}_{12}\hat{D}_{12}^T)\hat{C}_1 & 0 & 0 \\ 0 & \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} \Xi_{12} \\
&= {\Xi_{21}}^T \begin{bmatrix} \gamma^{-2}(sI + \hat{A}^T) & 0 & 0 & \\ 0 & -sI + \hat{A} & \hat{B}_1 & \hat{B}_2 \\ 0 & \hat{C}_1 & 0 & \hat{D}_{12} \\ 0 & \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} \Xi_{12}.
\end{aligned}
$$

Using the extended matrix pencil framework express in terms of the original state space matrices

$$-sE_k + A_k \quad = \quad \hat{\Xi}_{21}^T \begin{bmatrix} 1/\gamma^2(sI + A^T) & 0 & 0 & 0 & 0 \\ 0 & -sI + A & B_1 & B_2 & 0 \\ 0 & C_1 & D_{11} & D_{12} & 0 \\ 0 & C_2 & D_{21} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & -1/\gamma^2({D_{11}}^T) \end{bmatrix} \hat{\Xi}_{12}.$$

Therefore, $[-sE_k + A_k] = \hat{\Xi}_{21}^T \Pi(s) \hat{\Xi}_{12}$. Q.E.D.