Stability Analysis and Robust Control Synthesis with Generalized Multipliers

by

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A Dissertation Presented to the
FACULTY OF THE GRADUATE SCHOOL
UNIVERSITY OF SOUTHERN CALIFORNIA
In Partial Fulfillment of the
Requirements for the Degree
DOCTOR OF PHILOSOPHY
(Electrical Engineering)

July 2002

To

My Parents,

for their encouragement, support.

and

My wife Chih-Kuang

and

My lovely daughters Yang-Yang and Sherry

for their encouragement, support and sacrifice.

Acknowledgements

I would like to express my indebtness and highly respect to my guidance committee chairman, Professor M.G. Safonov, for his continuous support, inspiration and patient in the research presented in this dissertation. I also thank Professors H. Flashner and P. Ioannou for serving on the dissertation committee.

Acknowledgements are also due to the financial support from the Chung-Cheng Institute of Technology, National Defense University and to my senior officers in the Department of Weapon System Engineering for their encouragement and valuable suggestions during my pretty long research. I am grateful for the valuable discussion with Chih-Hung Huang, Gwo-Rei Yu and Tom Tsao during my research.

Friends and the members of softball team of the Chinese Culture Club bring me a good time during my study at the University of Southern California. In particular, those friends that work together during the Taiwan Film Festival did give me an unforgettable experience and memory.

Finally, I thank my family, especially my parents, for their sacrifice, love and continuous encouragement. Also, many thanks go to the family of my parents-in-law for their many get together dinners. Of course, I would like to give my deepest thanks to my wife for taking care our two lovely daughters while I was not at home during the Ph.D. study.

This Ph.D. study is really a long study. Thanks god, I finally finish it!!

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Abstract

The research in this dissertation mainly focuses on stability analysis and robust control synthesis with generalized multipliers. For stability analysis with generalized multipliers, we first link the conic sectors of the topological separation framework to the integral quadratic constraints of the IQC framework. The forms of quadratic separating functionals used in both the topological separation framework and integral quadratic constraints (IQCs) framework for stability analysis are presented.

Next, we examine the quadratic functionals used for topological separation in the IQC framework for robust stability analysis. A canonical factorization result is used to establish the existence of a stable minimum-phase factorization of these IQCs. The factors are constructed via the solution to a Riccati equation, and the IQCs are proved to be always unimodularly congruent to a J-matrix. The Riccati equation result provides a method to construct the invertible causal and anticausal factors of the multipliers that arise in the classical multiplier stability theory.

For robust control synthesis with generalized multipliers, we formulate a new unstably-weighted robust control synthesis problem. In this problem formulation, we obtain a robust controller without having to factorize the generalized multipliers, i.e., the unstable-weighting matrices. Based on the positive real control approach, we present Riccati inequalities and Riccati equations solution for the unstably-weighted control synthesis problem.

Chapter 1

Introduction

1.1 Introduction

This dissertation records the research conducted by the author on the stability analysis and robust control synthesis with generalized multipliers.

The research of this dissertation first shows that the quadratic separating functionals used in both the topological separation framework and integral quadratic constraints (IQCs) framework for stability analysis can be unimodularly congruent to an indefinite constant matrix. Next, under the multiplier approach the generalized multiplier IQCs can be unimodularly congruent to a positivity form with stable, minimum phase factors. This factorization results linked the generalized multiplier IQCs to the classical multiplier theory in passivity theorem.

On the other hand, the robust control synthesis based on the generalized multipliers approach can be shown to become the unstably-weighted robust control synthesis problem. The author presents a two-Riccati equation solution to solve the unstably-weighted control synthesis problem.

1.2 Stability Analysis with Generalized Multipliers

In the early 1960s, the principle of partitioning the open loop into two positive operators was introduced in the stability analysis of time-varying nonlinear feedback system. See [Pop62, Yak67, Zam66, Wil72, DV75, Saf80] and the references therein. The two most famous fundamental results in stability theory are small gain theorem and passivity theorem. However, both theorems deal with general classes of operators. Hence, in situations where more information about the operators is available, these theorems possibly yield conservative results. This difficulty can in theory be partially overcome by scalings in the small gain framework, and by multiplier theory in the passivity approach. Therefore, the use of multipliers in stability analysis with the passivity theorem can generally reduce conservatism of the analysis cosiderably.

In general, the multiplier M and its inverse are assumed to be bounded but not necessarily causal. However, the passivity theorem requires causal operators in the feedback interconnection and it can therefore not be applied to the system if M or M^{-1} is noncausal. In this case it is required that there exists a canonical factorization $M = M_+M_-$, where M_-, M_+^* and their inverses are causal and bounded. If such a factorization exists, the stability conditions of the original feedback systems can be preserved and less conservative analysis results can be obtained.

With the introduction of integral quadratic constraints (IQCs) by Megretski *et al.* [RM94, MR97], it was found that the multipliers can be introduced in a less restrictive manner under the IQC framework, without the causality constraints. On the other hand, Goh *et al.* [GS95] have shown that integral quadratic constraints are useful because they may be used to define the conic sectors which may be used to establish topological separation. They also showed that the quadratic constraints used for stability analysis must satisfy a congruent condition. However, the way to construct a stable factor in the congruent condition was not mentioned in [GS95].

Motivated by the use of generalized multipliers in the IQC framework, the research in this dissertation is focused on search of stable factorization forms that are necessary for establishing topological separation. We apply the canonical factorization techniques and show that the generalized multiplier IQCs can always satisfy a unimodularly congruent condition and have a stable, minimum phase factor in a diagonal form. It is this stable factorization that links the generalized multiplier IQCs to the classical multiplier theory and the passivity theorem.

1.3 Robust Control Synthesis with Generalized Multipliers

On the other hand, robust controller synthesis is the study of how control laws may be designed to maximize the robustness of a complex physical system under large modeling uncertainties, parameter variations and worst case conditions. In the past few decades we have seen tremendous advances in the design of robustly uncertainty tolerant multivariable feedback systems.

Since the introduction of the structured singular value, μ, by Doyle [Doy82] and the multivariable stability margin, k_m, by Safonov [Saf82], the robust control problems for real/complex parametric uncertainty have been studied by many. Transforming essentially from robust analysis theory, Safonov and Doyle [Saf83, Doy83] introduced the μ/k_m-synthesis approach for robust control synthesis. Their synthesis approaches are based on the $D-K$ iteration, a hybrid H_∞ control theory [DGKF89, SLC89] and the diagonal scaling techniques for multivariable stability margin analysis. However, one major drawback with the $D-K$ iteration approach is that it requires a curve fitting approximation after each D iteration which can significantly increase the designer's workload during the controller synthesis. To improve this approach, Fan *et al.* [FTD91] included real parametric uncertainty and Safonov *et al.* [SC93, SLC93] proposed an $M-K$ iteration that eliminates the need for curve fitting. Moreover, Balakrishnan [Bal95] unified these approaches by providing a sufficient condition for robust stability based on the passivity theorem and multipliers. This work also showed that the robust stability tests can be performed using convex optimization over *Linear Matrix Inequalities* (LMI's).

In [SC93], an $M-K$ iteration multiplier formulation of the μ/k_m-synthesis problem introduced in [CS92] was adopted. In this formulation the usual diagonal scalings are replaced with complex diagonal multipliers acting on a positive-real, bilinearly-transformed system. This multiplier formulation includes the diagonal scalings as a special case, but it also has the advantage that it is capable of producing less conservative μ/k_m-synthesis control law designs for the case in which some or all of the uncertainties are known to be real. The advantages of the multiplier perspective pave the way for a reliable, fully-automated μ/k_m-synthesis procedure. However, while working on the $M-K$ iteration, there is an intermediate step that required to compute the factorization of the noncausal (i.e., unstable) multiplier $M(s) = (M_2^T(-s))^{-1}M_1(s)$ where $M_1(s)$ and $M_2(s)$ and their inverses are stable and minimum phase. This factorization step certainly introduces computational complexity and the potential for numerical instability.

In this dissertation, based on the $M-K$ iteration we formulate a new unstably-weighted robust control synthesis problem: Given an unstable matrix $M(s)$, find a controller such that the unstably-weighted closed-loop system $M(s)T(s)$ is generalized strongly positive real and the closed-loop system $T(s)$ is stable. The presently available solutions are based on the positive real control approach. Therefore, these results bring a new direction to enable direct design of a robust controller without applying multiplier factorization.

1.4 Outline of this Dissertation

An outline of this dissertation is as follows.

- Chapter 2 analyses the role of quadratic separating functionals used in both the topological separation framework and integral quadratic constraints (IQCs) framework for stability analysis. The forms of quadratic separating functionals that are useful for establishing topological separation are presented.

- Working from the quadratic separating functionals used for stability and robustness analysis of the feedback systems, Chapter 3 shows that a stable, minimum phase state space factors always exists for the quadratic constraints in the topological framework and IQC framework for robust analysis. As a consequence, the existence of stable and stability invertible sector transformation form of spectral factors is proved.

- Chapter 4 consider the generalized multipliers used in IQC framework. We first showed the existence of canonical factorization of the generalized positive real (GPR) multipliers. Based on the canonical factorization results, we showed that the quadratic constraints with the generalized positive real multipliers in the IQC framework can be unimodularly congruent to an indefinite constant matrix and the stable, minimum phase factor has a diagonal form.

- Chapter 5 formulates a new unstably-weighted robust control synthesis problem. Based on the positive real control approach, we present a two-Riccati equation solution to the unstably-weighted control synthesis problem.

- Chapter 6 concludes this dissertation with a summary of the results presented and a proposal for future research.

Chapter 2

Quadratic Separating Functionals in Robust Analysis

2.1 Introduction

The robust analysis problem is in essence the problem of establishing a topological separation between the graph of a linear nominal plant and the set of inverse graphs of all possible plant uncertainty, see, e.g., [Zam66, Saf80], and more generally the works of [Pop62, San64, Wil71, DV75]. In the topological separation framework, conic sectors are important because they may be used to separate the space within which the graphs of operators lie. On the other hand, [RM94, MR97] introduce the integral quadratic constraints (IQCs) framework that gives renewed interest in stability analysis.

In this chapter, we examine the role of quadratic separating functionals which form the basis for both the conic sectors in the topological separation framework and the integral quadratic constraints of the IQC framework. It is shown in this chapter that the only integral quadratic constraints of interest for robustness analysis are those which are suitable for defining conic sectors, which will in turn establish the topological separation required for robust analysis. We will show that these quadratic constraints must satisfy a congruence condition to be useful as quadratic separating functional for establishing the topological separation.

2.2 Preliminaries

Notation used in this chapter and later chapters is summarized in Table 2.1. In the

Table 2.1: Notation

Symbol	Meaning
$\mathbf{R}$,($\mathbf{R}_+$)	Set of all (positive) real numbers
$\mathbf{C}^{oe}$	The extended $j\omega$ axis
$\overline{\mathbf{C}^+}$	The closed right half complex plane
$A^*(s)$, $A^\sim(s)$	$=A^T(-s)$, conjugate transpose
A^{-*}	$=(A^*)^{-1}$
I_q	$q \times q$ identity matrix
$\hat{x}(j\omega)$	Fourier transform of the signal $x(t)$, $\hat{x}(j\omega) = \int_0^\infty e^{-j\omega t}x(t)dt$
$< x, y >$	$=\int_{-\infty}^{\infty} y^T(t)x(t)dt$ $=\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}^*(j\omega)\hat{x}(j\omega)d\omega$
$\|x\|$	$=\sqrt{< x, x >}$
$\mathbf{RL}_\infty$	The set of proper (bounded at infinity) rational function with real coefficients
$\mathbf{RH}_\infty$	Subset of $\mathbf{RL}_\infty$ consisting of functions without poles in the closed right-half plane
$\mathbf{L}^l_{2,[0,\infty)}$	The space of $\mathbf{R}^l$-valued functions $f : [0, \infty) \mapsto \mathbf{R}^l$ of finite energy $\|f\|^2 = \int_0^\infty \|f(t)\|^2\, dt$
$\mathbf{L}^l_{2e,[0,\infty)}$	The members only need to be square integrable on finite intervals
spec(X)	The set of eigenvalues (spectrum) of matrix X

input-output theory for stability analysis we represent the systems as operators and their input and output signals as function from appropriate vector spaces. We begin by noting down some essential terms first.

Definition 2.1. *A normed vector space $\mathcal{L}$ is a linear vector space with a norm. The norm on $\mathcal{L}$ is a function $\|\cdot\| : \mathcal{L}$ (i.e., a nonnegative functional) that satisfies the properties*

1. $\|f\| = 0 \Leftrightarrow f \equiv 0$,
2. $\|\alpha f\| = |\alpha| \cdot \|f\|$,
3. $\|f + g\| \leq \|f\| + \|g\|$.

Remark 2.1. *The most frequently appearing function spaces in control applications are the l_p and $\boldsymbol{L}_p$ spaces, $p \geq 1$. We only consider the continuous time spaces $\boldsymbol{L}_p$ in this thesis. It consists of functions defined on the real axis. The vector spaces $\boldsymbol{L}_{p,[0,\infty)}$ consists of functions $f : \boldsymbol{R}_+ \to \boldsymbol{R}$ with norms*

$$\begin{array}{ll} \|f\|_p = (\int_0^\infty |f|^p \, dt)^{1/p} & \boldsymbol{L}_{p,[0,\infty)}, p = 1, 2, \ldots \\ \|f\|_\infty = ess \sup_{t \in \boldsymbol{R}_+} |f(t)| & \boldsymbol{L}_{\infty,[0,\infty)}. \end{array}$$

Remark 2.2. *We often need to use vector valued functions. We use the notation $\boldsymbol{L}^m_{p,[0,\infty)}$ to denote the functions $f : \boldsymbol{R}_+ \to \boldsymbol{R}^m$ with norm defined as above where now the spatial norm is the Euclidean norm $|f| = \left(f^T f\right)^{1/2}$.*

Definition 2.2. *An inner product space is a vector space with an inner product and the norm on the space can be defined in terms of inner product as*

$$\|f\| = \sqrt{\langle f, f \rangle}.$$

Remark 2.3. *All inner product spaces considered below are complete, i.e., their Cauchy sequences converge. Complete inner product spaces are called* Hilbert spaces *and will denote by* $\mathcal{H}$ *in order to distinguish their special structure from the normed vector spaces* $\mathcal{L}$*. The Hilbert spaces* $\boldsymbol{L}_{2,[0,\infty)}^m$ *has inner product defined as*

$$\langle f, g\rangle = \int_0^\infty f(t)^T g(t)dt = \tfrac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(j\omega)^* \hat{g}(j\omega)d\omega \qquad \boldsymbol{L}_{2,[0,\infty)}^m. \tag{2.1}$$

Definition 2.3. *An operator* H *is a mapping from one normed vector space into another. It is denoted as* $H : \mathcal{L} \to \mathcal{L}$.

Definition 2.4. *An operator* $H : \mathcal{L} \to \mathcal{L}$ *is linear if for any* α, $\beta \in \boldsymbol{R}$ *and functions* f, $g \in \mathcal{L}$ *such that*

$$H(\alpha f + \beta g) = \alpha H(f) + \beta H(g).$$

Remark 2.4. *We often use the shorthand notation* $G(f) = Gf$ *for the mapping of a linear operator* G.

Definition 2.5. *An operator* $H : \mathcal{L} \to \mathcal{L}$ *is called bounded if the following gain is finite*

$$\|H\| = \sup_{f\in\mathcal{L},\, f\neq 0} \frac{\|H(f)\|}{\|f\|}.$$

Definition 2.6. *The truncation operator* P_τ *is defines as follows.*

$$(P_\tau f)(t) = \begin{cases} f(t), & 0 \leq t \leq \tau, \\ 0, & t > \tau. \end{cases}$$

Remark 2.5. *We will often use the notation* $f_\tau = P_\tau f$.

Definition 2.7. *The extended space* $\mathcal{L}_e$ *is defined as*

$$\mathcal{L}_e = \{f : \|f_\tau\| < \infty,\ \forall \tau \geq 0\}.$$

Remark 2.6. *The extended space is an extension of a normed vector space consists of signals that may not be bounded in the norm of the vector space but where any truncation to a finite time intervals is bounded.*

Definition 2.8. *An operator* $H : \mathcal{L}_e \to \mathcal{L}_e$ *(or* $H : \mathcal{L} \to \mathcal{L}$*) is said to be causal (nonanticipative) if*

$$P_\tau H P_\tau = P_\tau H, \text{ for all} \tau \geq 0.$$

Remark 2.7. *The causality of operators on extended spaces means that the value at a certain time instant does not depend on future values of the argument. We note that* $H(f_\tau)(t) = H(f)(t)$ *when* $t \leq \tau$. *In other words, it does not matter if we truncate the future of the input signal when considering the output at a certain time instant.*

Definition 2.9. *A causal operator* $H : \mathcal{L}_e \to \mathcal{L}_e$ *is bounded if the gain defined as*

$$\|H\| = \sup_{f \in \mathcal{L},\, f \neq 0} \frac{\|H(f)\|}{\|f\|}$$

.

Remark 2.8. *It is clear that a bounded causal operator on* $\mathcal{L}_e$ *is also a well defined bounded causal operator on* $\mathcal{L}$. *This follows since if* $f \in \mathcal{L}$ *then* $\|P_\tau \Pi(f)\| \leq \|\Pi\| \cdot \|f\|$ *for all* $\tau \geq 0$. *We also have the reverse implication: A bounded causal operator on* $\mathcal{L}$ *is also a well defined bounded causal operator on* $\mathcal{L}_e$, *because* $P_\tau H(u) = P_\tau H(u_\tau)$, *and* $u_\tau \in \mathcal{L}$. *Thus, we have the following result*

$$H \text{ is causal and bounded on } \mathcal{L}_e \Leftrightarrow H \text{ is causal and bounded on } \mathcal{L}$$

.

Definition 2.10. *A causal operator G from $\boldsymbol{L}^p_{2e,[0,\infty)}$ to $\boldsymbol{L}^q_{2e,[0,\infty)}$ will be said to be* stable *if there exists a $\gamma \in \boldsymbol{R}_+$ such that $\|(Gu)_\tau\| \leq \gamma \|u_\tau\|$, $\forall \tau \geq 0$, for all $u \in \boldsymbol{L}^p_{2e,[0,\infty)}$.*

Definition 2.11. *Let $H : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator. Then the Hilbert adjoint H^* of H is the operator $H^* : \mathcal{H} \to \mathcal{H}$ such that*

$$\langle Hf, g\rangle = \langle f, H^* g\rangle \quad \forall f, g \in \mathcal{H}$$

.

Definition 2.12. *A bounded linear operator $H : \mathcal{H} \to \mathcal{H}$ is self-adjoint if $H^* = H$.*

Remark 2.9. *A bounded self-adjoint operator $\Pi = \Pi^* : \mathcal{H} \to \mathcal{L}$ defines a (bounded) quadratic form $\sigma : \mathcal{L} \to \boldsymbol{R}$ as $\sigma(f) = \langle \Pi f, f\rangle$. The quadratic form is positive semi-definite if $\sigma(f) \geq 0$ for all $f \in \mathcal{H}$ and strictly positive definite if there exists $\epsilon > 0$ such that $\sigma(f) > \epsilon \|f\|^2$, for all $f \in \mathcal{H}$.*

2.3 Topological Separation Framework

Much of the robustness analysis is rooted in the absolute stability results of [Pop62, San64, Zam66, Wil71, DV75]. They introduced a stability theory for input-output problems using functional analysis methods. The papers of [Zam66] provided useful conditions that could be imposed on the open-loop behavior of the feedback components of an interconnection to guarantee stability of the closed loop. The fundamental result was that "if the open loop can be factored into two suitably proportioned conic relations, then the closed loop is bounded." The well-known small gain theorem and passivity theorem were special cases. In particular, one may note

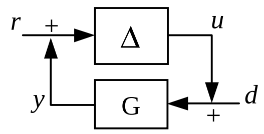

Figure 2.1: Connection of Nominal Plant and Uncertainty

that the work by Safonov [Saf80], building on the framework of Zames [Zam66], gives an interpretation of absolute stability criteria in terms of the requirement for *topological separation* between the inverse graphs of the plant uncertainty set and the graph of nominal plant.

In this section, we first present the use of separating functionals to establish topological separation. Then the conic sectors for topological separation and their connection to quadratic functionals are introduced.

2.3.1 Separating functionals and Topological Separation

Consider the feedback system in Figure 2.1 where G is a n_u-input n_y-output stable, causal operator from $L^{n_u}_{2e,[0,\infty)}$ to $L^{n_y}_{2e,[0,\infty)}$, and Δ is a n_y-input n_u-output stable, causal operator from $L^{n_y}_{2e,[0,\infty)}$ to $L^{n_u}_{2e,[0,\infty)}$. For the robustness analysis purposes, the operator G usually represents the nominal plant (or the nominal closed loop in robust controller synthesis), and the operator Δ represents an element of the plant uncertainty set, $\mathcal{D}$.

According to Definition 2.10, the interconnection of Figure 2.1 is said to be *stable* if there exists a $\gamma \in \mathbf{R}_+$ such that

$$\left\| \begin{bmatrix} u \\ y \end{bmatrix}_\tau \right\| \leq \gamma \left\| \begin{bmatrix} r \\ d \end{bmatrix}_\tau \right\|, \ \forall \tau \geq 0,$$

and for all $\begin{bmatrix} r \\ d \end{bmatrix} \in \mathbf{L}_{2e,[0,\infty)}^{n_y+n_u}$. See, e.g., [Saf80, Vid92], for more details.

The stable operators G and Δ each define *relations* or *graphs* in the space $\mathbf{L}_{2e,[0,\infty)}^{n_u+n_y}$, i.e., the subsets of $\mathbf{L}_{2e,[0,\infty)}^{n_u+n_y}$ defined by

$$\mathcal{G}_G := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{L}_{2e,[0,\infty)}^{n_u+n_y} : y = Gu \right\}, \tag{2.2}$$

$$\mathcal{G}_\Delta := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathbf{L}_{2e,[0,\infty)}^{n_u+n_y} : u = \Delta y \right\}. \tag{2.3}$$

Define also the *inverse graph* of Δ,

$$\mathcal{G}_\Delta^I := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{L}_{2e,[0,\infty)}^{n_u+n_y} : u = \Delta y \right\}. \tag{2.4}$$

Based on those defined graphs, [Saf80] states that the interconnection of Figure 2.1 is stable if the graphs $\mathcal{G}_G$ and $\mathcal{G}_\Delta^I$ intersect only at the origin, i.e., if a *topological separation* exists between the graph $\mathcal{G}_G$ and the inverse graph $\mathcal{G}_\Delta^I$. Therefore, in order to separate the graphs $\mathcal{G}_G$ and $\mathcal{G}_\Delta^I$, we need to find a *separating functional* to establish the topological separation. See [Saf80, Chapter 2] for further details, and also [Tee96].

It is shown in [Saf80, Chapter 2] and [Zam66] that a sufficient condition for topological separation is established if for every $\tau \in [0,\infty)$, there exists a separating functional, $d_\tau : \mathbf{L}_{2e,[0,\infty)}^{n_u+n_y} \mapsto \mathbf{R}$, and $\epsilon > 0$ such that

$$\begin{aligned} d_\tau(u,y) &> \epsilon\left(\|u_\tau\|^2 + \|y_\tau\|^2\right), \forall (u,y) \in \mathcal{G}_G \\ d_\tau(u,y) &\leq 0, \forall (u,y) \in \mathcal{G}_\Delta^I. \end{aligned} \tag{2.5}$$

Thus, if we can find a suitable function satisfies the separation conditions of (2.5), topological separation for the two graphs $\mathcal{G}_G$ and $\mathcal{G}_\Delta^I$ will be established and so is the stability of the interconnection system.

In the following section, we will connect the concept of conic sectors to the separating functionals within the topological separation framework. Furthermore, we will show the quadratic forms that are integral in the definition of conic sectors play an important role in the topological separation framework.

2.3.2 Conic Sectors for Topological Separation

The concept of the conic sector is particularly useful for stability analysis, see, e.g., [Vid92, page 221] and [Zam66, Saf80]. In essence, conic sectors are used to prove a topological separation of the graph and the inverse graph of a pair of operators, and hence stability.

In robust analysis, it is desired to establish stability of the system in Figure 2.1, when G is fixed, and as Δ varies over $\mathcal{D}$. Therefore the most common approach is to attempt to find a separating functional $d_\tau(u, y)$ to establish a topological separation between the graph $\mathcal{G}_G$ and the set $\left\{\mathcal{G}_\Delta^I : \Delta \in \mathcal{D}\right\}$. Usually, the set $\left\{\mathcal{G}_\Delta^I : \Delta \in \mathcal{D}\right\}$ is a priori assumed to be located within the intersection of a family of *conic sectors*. Then robustness analysis is limited to finding a conic sector from that family for which $\mathcal{G}_G$ lies outside the set of plant uncertainty graph. Therefore, the attempt to find a separating functional to establish a topological separation becomes finding a conic sector to accomplish the separating of two graphs of the interconnection system.

There are several ways one may define a conic sector. The following definitions are found in [Shi74, page 228-231], [Vid92, page 221] and [Zam66], respectively.

$$\mathcal{S}_H := \left\{x \in \mathbf{R}^n : x^* H x \le 0, H = H^T\right\} \tag{2.6}$$

$$\begin{aligned}
\mathcal{S}_{A,B} &:= \left\{\left[x^T y^T\right]^T, x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, n_1 + n_2 = n : \langle y - Ax, Bx - y\rangle \le 0\right\} && (2.7)\\
&= \Bigg\{z = \left[x^T y^T\right]^T, x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, n_1 + n_2 = n : \\
&\qquad z^T \begin{bmatrix} -(A^T B + B^T A) & (A+B)^T \\ A+B & -2I \end{bmatrix} z \le 0 \Bigg\}, && (2.8)
\end{aligned}$$

$$\begin{aligned}
\mathcal{S}_{M,C} &:= \left\{\left[x^T y^T\right]^T, x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, n_1 + n_2 = n : \|y - Cx\| \le \|Mx\|\right\} && (2.9)\\
&= \Bigg\{z = \left[x^T y^T\right]^T, x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, n_1 + n_2 = n : \\
&\qquad z^T \begin{bmatrix} C^T C - M^T M & -C \\ -C^T & I \end{bmatrix} z \le 0 \Bigg\}. && (2.10)
\end{aligned}$$

In all three cases, it may be seen that conic sectors may all be described using a *quadratic form.* Moreover, the first case is actually more general than the other two cases, which are indefinite under the mild conditions that $A - B$ and M are non-singular, respectively.

In general, given symmetric matrix $H \in \mathbf{R}^{(n_u+n_y)\times(n_u+n_y)}$, $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$, with $H_{11} \in \mathbf{R}^{n_u \times n_u}$ and $H_{22} \in \mathbf{R}^{n_y \times n_y}$, and a quadratic functional $q : \mathbf{R}^{n_u+n_y} \mapsto \mathbf{R}$ given by

$$q\left(\begin{bmatrix} u \\ y \end{bmatrix}\right) := \begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}, \tag{2.11}$$

one may define a conic sector $\mathcal{S} \subset \mathbf{R}^{n_u+n_y}$ by,

$$\mathcal{S} := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{R}^{n_u+n_y}, u \in \mathbf{R}^{n_u}, y \in \mathbf{R}^{n_y} : \right.$$
$$\left. q\left(\begin{bmatrix} u \\ y \end{bmatrix}\right) := \begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \leq 0 \right\}, \tag{2.12}$$

and for any $\epsilon > 0$ the ϵ-complement conic sector of $\mathcal{S}$ is defined by,

$$\mathcal{S}^{c,\epsilon} = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{R}^{n_u+n_y} : q\left(\begin{bmatrix} u \\ y \end{bmatrix}\right) \geq \epsilon \left\|u\right\|^2 + \left\|y\right\|^2 \right\}. \tag{2.13}$$

Note that $\mathcal{S} \cap \mathcal{S}^{c,\epsilon} = 0$.

From the definition of conic sectors, we can see that the important of conic sectors is that they seem to be the simplest forms for describing sets containing graphs of operators, or sets of operators. Similarly, quadratic forms, which are already integral to the way conic sectors are defined, also seem to be the simplest functions for establishing topological separation between graphs of one set of operators and the inverse graphs of another set of operators.

Let us consider now linear operators (i.e., matrices) G from $\mathbf{C}^{n_u}$ to $\mathbf{C}^{n_y}$, and Δ from $\mathbf{C}^{n_y}$ to $\mathbf{C}^{n_u}$. It is shown in [GS95] that if there exists a hermitian matrix $H \in \mathbf{C}^{n_u+n_y}$, and an $\epsilon > 0$ such that for the conic sector of (2.12), for some matrix G and all matrices $\Delta \in \mathcal{D}$,

$$\mathcal{G}_G \subset \mathcal{S}^{c,\epsilon}, \text{and } \mathcal{G}_\Delta^I \subset \mathcal{S}. \tag{2.14}$$

then it may easily shown that the matrices $I_{n_u} - \Delta G$ and $I_{n_y} - G\Delta$ are invertible for all $\Delta \in \mathcal{D}$.

If we consider the interconnection of Figure 2.1, it is obvious that a necessary and sufficient condition for the stability of interconnection for all $\Delta \in \mathcal{D}$ is for the transfer function $I_{n_u} - \Delta G$ and $I_{n_y} - G\Delta$ to be invertible for all $\Delta \in \mathcal{D}$ and all $s \in \overline{\mathbf{C}^+}$. Therefore, the separation condition of (2.14) will guarantee the stability of the feedback system.

In summary, the conic sectors are important because they may be used to define sets within spaces containing of graphs of operator. The quadratic functionals are in turn important because they may be used to define conic sectors. Therefore, the quadratic separating functional plays an important role within the topological separation framework.

2.4 Intergral Quadratic Constraints Framework

Seeking methods to cope with uncertainties or nonlinearities has been a major aim of robust control theory. Recently, many of methods that have been developed within the area of robust control have been reformulated within the framework of integral quadratic constraints (IQCs) [RM94, MR97].

The IQC framework did not appear from nowhere. It has its roots in at least three strong research fields: The input-output theory developed by Zames, Sandberg, Williems and many others [Zam66, ZF68, Wil71, San65b, San65a, DV75, Saf80], the absolute stability theory with extraordinary contributions from Yakubovich and Popov [Yak62, Yak63, Yak65b, Yak65a, Yak67, Yak71, Yak82, Pop61], and finally the robust control field with contributions from, for example, Zames, Safonov, Doyle, and many others [Saf82, Doy82, FTD91, BDG+93, PD93, ZDG96].

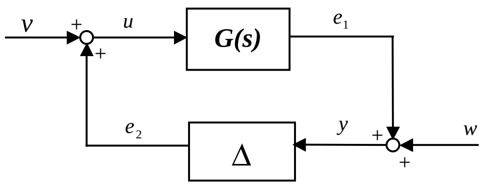

Figure 2.2: Block diagram of the feedback system

In this section, the connection between the topological separation framework and integral quadratic constraints framework will be stated. In particular, it is shown that the quadratic separating functional also plays an important roles in the IQCs framework.

Consider the feedback system of Figure 2.2 where G and Δ are bounded causal operators on $\mathbf{L}^m_{2e,[0,\infty)}$ and $\mathbf{L}^l_{2e,[0,\infty)}$, respectively. In the following, the definition of well-posed system and stable feedback system will first be described.

Definition 2.13. *[MR97] The interconnection G and Δ is said to be well-posed if the map $(y, u) \mapsto (v, w)$ has a causal inverse on $\boldsymbol{L}^{l+m}_{2e,[0,\infty)}$. The feedback system is said to be stable if it is well-posed and inputs $v \in \boldsymbol{L}^m_{2,[0,\infty)}$, $w \in \boldsymbol{L}^l_{2,[0,\infty)}$ lead to outputs e_1, $y \in \boldsymbol{L}^l_{2,[0,\infty)}$ and e_2, $u \in \boldsymbol{L}^m_{2,[0,\infty)}$. If, in addition, there exists a constant $C > 0$ such that*

$$\int_0^T \left(|y|^2 + |u|^2\right) dt \leq C \int_0^T \left(|v|^2 + |w|^2\right) dt, \ \forall T \geq 0,$$

then, the system is said to be stable with finite gain.

Next, a formal definition of the term IQC can be made as follows.

Definition 2.14. *[RM94] Suppose* $\Pi : j\boldsymbol{R} \mapsto \boldsymbol{C}^{(l+m)\times(l+m)}$ *is a bounded measurable function taking Hermitian values. Let* σ *be the quadratic form defined on* $\boldsymbol{L}^l_{2,[0,\infty)} \times \boldsymbol{L}^m_{2,[0,\infty)}$ *by*

$$\sigma(y,u) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \hat{u}(j\omega) \end{bmatrix} d\omega. \tag{2.15}$$

A bounded causal map $\Delta : \boldsymbol{L}^l_{2,[0,\infty)} \mapsto \boldsymbol{L}^m_{2,[0,\infty)}$ *is said to satisfy the IQC defined by* Π*, also written* $\Delta \in IQC(\Pi)$*, if*

$$\sigma(y, \Delta y) \geq 0 \;\; \forall y \in \boldsymbol{L}^l_{2,[0,\infty)}. \tag{2.16}$$

From the definition of IQC, we can see that IQC provide a way of representing relationships between processes evolving in a complex dynamical system, in a form that is convenient for analysis. Moreover, the IQC are defined in terms of quadratic forms which are defined in terms of self-adjoint operators (i.e., $\Pi(j\omega) = \Pi^*(j\omega)$). Depending on particular application, various versions of IQC's are available. Now, a general stability theorem in terms of IQC can be stated as follows.

Theorem 2.1. *[MR97, Theorem 1] Assume that:*

1. *for every* $\alpha \in [0,1]$*, the interconnection of* G *and* Δ_α *is well-posed where* Δ_α *is a parameterization of* Δ *which satisfies*

   (a) $\Delta = \Delta|_{\alpha=1}$*,*

   (b) Δ_α *is bounded and causal for* $\alpha \in [0,1]$*,*

   (c) there exists $\gamma > 0$ *such that*

$$\||\Delta_{\alpha 1}(y) - \Delta_{\alpha 2}(y)\| \leq \gamma\, |\alpha_1 - \alpha_2| \cdot \|y\|$$

for all α_1, $\alpha_2 \in [0,1]$,

2. *the interconnection of* G *and* $\Delta_\alpha|_{\alpha=0}$ *is stable with finite gain,*

3. *for every* $\alpha \in [0,1]$, *the IQC defined by* Π *is satisfied by* Δ_α, *that is,*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{\Delta}(y)(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \hat{\Delta}(y)(j\omega) \end{bmatrix} d\omega \geq 0 \tag{2.17}$$

4. *there exists* $\epsilon > 0$ *such that*

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \forall \omega \in \boldsymbol{R}. \tag{2.18}$$

Then , the feedback interconnection of G *and* Δ *is stable with finite gain for all* $\alpha \in [0,1]$.

Remark 2.10. *If we compare the definition of conic sector of (2.12) and the definition of IQC of (2.15), we can notice that the signals* u *and* y *stacked in reverse order. Indeed, we can reverse the order of these signals, but the corresponding forms of (2.17) and (2.18) need to be changed accordingly.*

Therefore, by the IQC definition and IQC stability theorem, [MR97] states that a sufficient condition for robust stability of the interconnection of Figure 2.2 over all Δ is the existence of some $\epsilon > 0$, and some $\Pi(j\omega) = \Pi^*(j\omega)$, such that

$$\begin{aligned} \sigma(u,y) &\geq \epsilon \left\|(u, Gu)\right\|^2, & \forall u \in \mathbf{L}^l_{2,[0,\infty)} \\ \sigma(\alpha u, y) &\leq 0, \forall \alpha \in [0,1] \text{and for all} \Delta & \forall y \in \mathbf{L}^m_{2,[0,\infty)} \end{aligned} \tag{2.19}$$

Hence, it is clear that the condition (2.19) could be expressed in terms of conic sectors, therefore the IQC framework also establishes topological separation of the graph G and the set of inverse graph of Δ in the operator space.

2.5 Form of Quadratic Separating Functionals

In the previous two sections, we have shown that the conic sectors in topological separation framework and the integral quadratic constraints in the IQC framework are all based on a quadratic form/functional to establish the stability of the feedback systems. We will show in this section that these quadratic functionals must satisfy a congruent condition to be useful as separating functionals.

In the following, we restrict attention to the case where G is a stable, causal, linear time invariant operator, and where every $\Delta \in \mathcal{D}$ is also a stable, causal, linear time invariant operator. In this case, it will be sufficient to consider graphs and quadratic separating functionals in $\mathbf{C}^{n_u+n_y}$ parameterized by frequency $j\omega$.

Let the stable, causal, linear time invariant operator G from $\mathbf{C}^{n_u}$ to $\mathbf{C}^{n_y}$ have Laplace transform of its impulse response given by $G(s)$. Let $\mathcal{D}$ be a set of (possible structured) stable, causal, linear time invariant operators from $\mathbf{C}^{n_y}$ to $\mathbf{C}^{n_u}$, and if $\Delta \in \mathcal{D}$, let the Laplace transform of its impulse response be given by $\Delta(s)$. It then follows that $G(s)$ and every $\Delta \in \mathcal{D}$ are bounded on $\overline{\mathbf{C}^{+}}$, and have corresponding Fourier transforms $G(j\omega)$ and $\Delta(j\omega)$.

For each fixed $j\omega$, define the graph of $G(j\omega)$ and the inverse graph of $\Delta(j\omega)$ in the space $\mathbf{C}^{n_u+n_y}$,

$$\begin{aligned} \mathcal{G}_{G(j\omega)} &:= \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{C}^{n_u+n_y} : y = G(j\omega)u \right\}, \\ \mathcal{G}^I_{\Delta(j\omega)} &:= \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{C}^{n_u+n_y} : u = \Delta(j\omega)y \right\}. \end{aligned} \tag{2.20}$$

Now, given that

$$H(j\omega) = H^*(j\omega) = \begin{bmatrix} H_{11}(j\omega) & H_{12}(j\omega) \\ H_{12}^*(j\omega) & H_{22}(j\omega) \end{bmatrix}$$

bounded and measurable over $\mathbf{C}^{oe}$, consider the frequency dependent quadratic functional on $\mathbf{C}^{n_u+n_y}$,

$$q(z) := z^* H(j\omega) z. \tag{2.21}$$

For each $j\omega$ fixed, one may define the following sector on the space $\mathbf{C}^{n_u+n_y}$, and its ϵ-complement,

$$\mathcal{S} := \{z : q(z) \leq 0\}, \tag{2.22}$$

$$\mathcal{S}^{c,\epsilon} := \left\{z : q(z) \geq \epsilon \|z\|^2\right\}. \tag{2.23}$$

Note that $\mathcal{S} \cap \mathcal{S}^{c,\epsilon} = 0$ for every $\epsilon > 0$.

Then, we can have the following key observation.

Proposition 2.1 ([GS95]). *The graph of G, $\mathcal{G}_G$, is an n_u-dimensional subspace of $\mathbf{C}^{n_u+n_y}$ given by Im$\left\{\begin{bmatrix} I_{u_u} \\ G \end{bmatrix}\right\}$. Further, for $\mathcal{S}$ and H of (2.22), if $\mathcal{G}_G \in \mathcal{S}^{c,\epsilon}$, then H has n_u eigenvalues $\geq \epsilon$.*

The inverse graph of Δ, $\mathcal{G}_\Delta^I$, *is an* n_y*-dimensional subspace of* $\boldsymbol{C}^{n_u+n_y}$ *given by* $Im\left\{\begin{bmatrix}\Delta \\ I_{n_y}\end{bmatrix}\right\}$. *Further, for* $\mathcal{S}$ *and* H *of (2.22), if* $\mathcal{G}_\Delta^I \in \mathcal{S}$, *then* H *has* n_y *eigenvalues* ≤ 0.

Proof. See [GS95].

□

On the other hand, the integral quadratic constraints, i.e., the frequency domain quadratic separating functionals, in the IQC framework can be written in the following form

$$d(z) := \int_{-\infty}^{\infty} \hat{z}^*(j\omega)\Pi(j\omega)\hat{z}(j\omega)d\omega \tag{2.24}$$

where $\Pi(j\omega) = \Pi^*(j\omega)$ is bounded and measurable over $\mathbf{C}^{oe}$. From the definition of IQC and its stability theorem, it can be shown that a sufficient condition for robust stability of the interconnection of Figure 2.1 over all $\Delta \in \mathcal{D}$ is the existence of some $\epsilon > 0$, and some $\Pi(j\omega) = \Pi^*(j\omega)$, bounded and measurable over $\mathbf{C}^{oe}$, such that

$$\begin{aligned} d\left(\begin{bmatrix} u \\ y \end{bmatrix}\right) &\geq \epsilon \left\|(u, Gu)\right\|^2 & \forall u \in \mathbf{C}^{n_u}, \\ d\left(\begin{bmatrix} \alpha u \\ y \end{bmatrix}\right) &\leq 0,\ \forall \alpha \in [0,1], \forall \Delta \in \mathcal{D} & \forall y \in \mathbf{C}^{n_u} \end{aligned} \tag{2.25}$$

The constraints (2.25) may be expressed in terms of sectors, and hence topological separation of the graph $G(j\omega)$ and inverse graph of $\Delta(j\omega)$ in $\mathbf{C}^{n_u+n_y}$ space is established. Therefore, the topological separation approach proposed by [Saf80] was renewed with the introduction of the so-called analysis via Integral Quadratic Constraints [MR97].

Note that the quadratic functional $q(z)$ in conic sectors and $d(z)$ in IQC framework do have a common function, i.e., $H(j\omega)$ and $\Pi(j\omega)$. In fact, they are the same function. It is this function that makes the quadratic functional becoming the quadratic separating functional. However, it is shown in [GS95] that the function $\Pi(j\omega)$ must satisfy a congruent condition in order to make quadratic functionals being useful as quadratic separating functionals.

Assume $\Pi(j\omega)$ is invertible over $\mathbf{C}^{oe}$. The following theorem shows the congruent condition for the $\Pi(j\omega)$ function to be useful as a separating functional.

Theorem 2.2 ([GS95]). *If the matrix transfer function* $\Pi(j\omega) = \Pi^*(j\omega)$, *continuous, bounded and invertible on* $\boldsymbol{C}^{oe}$, *is such that (2.25) holds, then there exists a* $S(j\omega)$, *bounded and invertible over* $\boldsymbol{C}^{oe}$ *such that*

$$\Pi(j\omega) = S^*(j\omega) J_{n_u,n_y} S(j\omega), \forall j\omega \in \boldsymbol{C}^{oe} \tag{2.26}$$

where $J_{n_u,n_y} := diag\left(I_{n_u,-I_{n_y}}\right)$.

Proof. Note that the first part of (2.25) implies that

$$\begin{bmatrix} I \\ G(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ G(j\omega) \end{bmatrix} > \epsilon I_{n_u}, \forall j\omega \in \mathbf{C}^{oe},$$

by, e.g., [MT93, Theorem 3.1] and the continuity of $\Pi(j\omega)$. Hence by Proposition 2.1 at each $j\omega \in \mathbf{C}^{oe}$, $\Pi(j\omega)$ has at least n_u positive eigenvalues. On the other hand, the second part of (2.25) implies that

$$\Pi_{22}(j\omega) \leq 0, \ \forall j\omega \in \mathbf{C}^{oe},$$

by, e.g., [MT93, Theorem 3.1] and the continuity of $\Pi(j\omega)$, and by setting $\alpha = 0$. Hence, since by assumption $\Pi(j\omega)$ is invertible over $\mathbf{C}^{oe}$, there exists a $S(j\omega)$, bounded and invertible over $\mathbf{C}^{oe}$ such that (2.26) holds.

□

Therefore, in order to be useful for robust control analysis, the hermitian function $\Pi(j\omega)$ of the quadratic functional $d(z)$ (or $H(j\omega)$ of $q(z)$) must necessarily have the form

$$\Pi(j\omega) = S^*(j\omega) \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix} S(j\omega)$$

where $S(j\omega)$ is a frequency dependent matrix with no poles or zeros on $\mathbf{C}^{oe}$.

Remark 2.11. *In general, without loss of generality, we can assume for simplicity that $n_u = n_y$. Therefore the congruent condition can be denoted as*

$$\Pi(j\omega) = S^*(j\omega) \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_u} \end{bmatrix} S(j\omega)$$

In the following chapters, we will use this form for our analysis.

Remark 2.12. *The vector* $\begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix} = S \begin{bmatrix} u \\ y \end{bmatrix}$ *may be said to belong to the graph of a sector transformed plant,*

$$\tilde{G}(j\omega) := (S_{21} + S_{22}G(j\omega))\,(S_{11} + S_{12}G(j\omega))^{-1}$$

if $\begin{bmatrix} u \\ y \end{bmatrix} \in \mathcal{G}_{G(j\omega)}$.

Remark 2.13. *The connections between IQC and stability results based on conic sectors of [Zam66, Saf80] are now briefly examined. Given*

$$S(j\omega) := \begin{bmatrix} S_{11}(j\omega) & S_{12}(j\omega) \\ S_{21}(j\omega) & S_{22}(j\omega) \end{bmatrix} \text{ and } z = \begin{bmatrix} u \\ y \end{bmatrix},$$

note that

$$\begin{aligned} d(z) &= \textstyle\int_{-\infty}^{\infty} z^*(j\omega)\Pi(j\omega)z(j\omega)d\omega \\ &= \textstyle\int_{-\infty}^{\infty} \begin{bmatrix} u \\ y \end{bmatrix}^* \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} d\omega \qquad (2.27) \\ &= \|S_{11}(j\omega)u + S_{12}(j\omega)y\|^2 - \|S_{21}(j\omega)u + S_{22}(j\omega)y\|^2 \end{aligned}$$

Hence, e.g., for the case where $S_{22}(s) = 0$, $\forall s \in \boldsymbol{C}$, $\begin{bmatrix} S_{21}(s) & S_{22}(s) \end{bmatrix}$ *is bounded and has full rank on* $\overline{\boldsymbol{C}^+}$ *and* $S_{11}(s)$ *is bounded and invertible on* $\overline{\boldsymbol{C}^+}$, *the constraints (2.25) becomes a conic type constraints, e.g.,*

$$\|(S_{22}Gu + S_{21}u)_\tau\|^2 \leq \|(S_{11}u)_\tau\|,$$

$\forall u \in \boldsymbol{L}_{2,[0,\infty)}^{n_u}$, $\forall \tau \in [0, \infty)$.

2.6 Summary

This chapter studies the quadratic functionals in both the conic sectors of the topological separation framework and the integral quadratic constraints of the IQC framework. The results of this chapter show that integral quadratic constraints are important because they may define the conic sectors for establishing topological separation. A further result of this chapter is the congruent condition for the quadratic functionals to become the separating functionals that are useful for topological separation. The next chapter will examine the stable factors of the quadratic constraints in robust control.

Chapter 3

Factorization of Quadratic Constraints in Robust Analysis

3.1 Introduction

In Chapter 2, we have linked the integral quadratic constraint (IQC) approach for robust analysis to previous work on generalized sectors, dissipativity and J-spectral factorization. The results of Chapter 2 show that quadratic functionals of IQC framework [RM94, MR97] used for robust analysis, i.e., $q(z) = \int_{-\infty}^{\infty} z^* \Pi(j\omega) z d\omega$, must have the form $\int_{-\infty}^{\infty} z^* S^*(j\omega) J S(j\omega) z d\omega$, for some invertible $S(j\omega)$ and $J = \operatorname{diag}(I, -I)$. In other words, J-spectral factors of IQCs must always exist. Therefore the topological separation of graphs can be tested using IQCs.

Moreover, in order to be easier to prove stability in the IQC framework, we usually need to have a stable factor from the factorization of the quadratic constraints. However, the way to construct the stable spectral factors was not mentioned in Chapter 2. In this chapter, we show that using the canonical factorization techniques the generalized rational IQC weighting function $\Pi(s)$ will satisfy a congruent condition with a stable factor. In fact, the result shows that $\Pi(S)$ is unimodularly congruent to an indefinite constant matrix.

3.2 Problem Formulation

Let us first introduce the concept of unimodularly congruence that is used in this chapter.

Consider the set of polynomials $\mathcal{R}[s]$ in s with real coefficients, together with the usual operations of addition, $+$, and multiplication, $\cdot$, of polynomials. Then $(\mathcal{R}[s], +, \cdot)$ is a *ring*. Therefore, $\mathcal{R}[s]$ denotes the ring of polynomials with real coefficient. Let $\mathcal{R}^{n\times m}[s]$ be the set of all $n \times m$ matrices with coefficients in $\mathcal{R}[s]$. An element of $\mathcal{R}^{n\times m}[s]$ is called a polynomial matrix.

Let $\mathcal{R}(s)$ denote the field of rational functions with real coefficients. Then, $\mathcal{R}^{n\times m}(s)$ is the set of all $n \times m$ matrices with coefficients in $\mathcal{R}(s)$. An element of $\mathcal{R}^{n\times m}(s)$ is called a rational matrix. A rational matrix G is called stable if G has no poles in the closed right half complex plane. G is called proper if $\lim_{s\to\infty} G(s)$ exists.

In the following, we will focus on the proper, stable and real-rational matrices in the problem formulation. The definition of unimodularly congruent that used in the problem formulation is defined as follows.

Definition 3.1. *A square real-rational matrix $U(s)$ is called* unimodular *if it has an inverse over the ring of proper, stable and real-rational matrices.*

Definition 3.2. *Two square real-rational matrices $P(s)$ and $Q(s)$ are called* unimodularly congruent *if a unimodular matrix $U(s)$ exists such that $Q(s) = U^{\sim}(s)P(s)U(s)$.*

Since we consider the ring of proper, stable and real-rational matrices, the unimodular matrix $U(s)$ has the property that $U(s)$ and $U^{-1}(s)$ are proper, stable and real-rational matrices, i.e., $U(s)$, $U^{-1}(s) \in \mathbf{RH}_\infty$. Therefore, if we apply rational matrices to transfer function matrices, the $U(s)$ would be a stable and minimum phase transfer function matrix. Hence, the unimodularly congruent concept in this chapter would imply the existence of a stable and minimum phase factor in the congruent condition.

Recall from the previous chapter, we have shown that if $\Pi(j\omega) = \Pi^*(j\omega)$ is invertible over $\mathbf{C}^{oe}$, the quadratic constraints

$$q\left(\begin{bmatrix} u \\ y \end{bmatrix}\right) = \int_{-\infty}^{\infty} \begin{bmatrix} \hat{u}(j\omega) \\ \hat{y}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{u}(j\omega) \\ \hat{y}(j\omega) \end{bmatrix} d\omega. \tag{3.1}$$

is only useful as separating functionals if there exists a $S(j\omega)$, bounded and invertible over $\mathbf{C}^{oe}$ such that

$$\Pi(j\omega) = S^*(j\omega) \begin{bmatrix} I_{n_u} & 0 \\ 0 & -I_{n_y} \end{bmatrix} S(j\omega), \quad \forall j\omega \in \mathbf{C}^{oe}. \tag{3.2}$$

In this chapter, we may assume without loss of generality that $n_u = n_y = n$ and there exists a para-Hermitian extension of $\Pi(j\omega)$ over $\mathbf{C}$, i.e., there exists $\Pi(s)$ such that $\Pi(s) = \Pi^\sim(s)$, $\forall s \in \mathbf{C}$ that obeying the same separating properties of $\Pi(j\omega)$. Therefore, we are interested in finding s stable and minimum phase transformation $S(s)$ for the general quadratic constraints that satisfying the congruent condition (3.2). Therefore, the problem of finding a stable factorization of quadratic constraints can be stated as follows.

Problem Formulation. *Given a continuous, bounded and invertible matrix transfer function* $\Pi(s) = \Pi^{\sim}(s) \in \boldsymbol{RL}_{\infty}$*, find a stable and minimum phase* $S(s)$ *such that*

$$\Pi(s) = S^{\sim}(s) \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} S(s), \forall s \in \boldsymbol{C}.$$

3.3 Canonical Factorization and Algebraic Riccati Equation

In order to establish the stable factorization of quadratic constraints, we will apply the technique of canonical factorization for the rational transfer function matrices.

Definition 3.3 (Canonical Factorization [Fra87, Goh96]). *Let a square matrix* $X(s)$ *has the property that* $X(s)$, $X^{-1}(s) \in \boldsymbol{RL}_{\infty}$. *Then* $X(s) = X_{+}(s)X_{-}(s)$, $\forall s \in \boldsymbol{C} \cup \{\infty\}$ *and* $X_{-}(s)$, $X_{-}^{-1}(s)$, $X_{+}^{\sim}(s)$, $(X_{+}^{-1}(s))^{\sim} \in \boldsymbol{RH}_{\infty}$, *is called a* canonical factorization.

Remark 3.1. *Since the factors* $X_{-}(s)$ *and* $X_{-}^{-1}(s) \in \boldsymbol{RH}_{\infty}$, *they are both proper, stable transfer function matrices. In other words, the factor* $X_{-}(s)$ *in the canonical factorization is a stable and minimum phase transfer function matrix.*

Let $\stackrel{ss}{=}$ denote a state-space representation of a real rational transfer function matrix, i.e., if

$$X(s) := D + C(sI - A)^{-1}B,$$

then

$$X(s) \stackrel{ss}{=} [A, B, C, D] \stackrel{ss}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \stackrel{ss}{=} \left[\begin{array}{c|c} T^{-1}AT & T^{-1}B \\ \hline CT & D \end{array}\right]$$

where the nonsingular matrix T defines an equivalence transformation. The poles of $X(s)$ are the eigenvalues of A matrix, and, if D is invertible, the zeros of $X(s)$ are the eigenvalues of $A - BD^{-1}C$. Finally, for any $X(s) = D_1 + C_1(sI - A_1)^{-1}B_1$ and $Y(s) = D_2 + C_2(sI - A_2)^{-1}B_2$ of compatible dimensions,

$$Y(s)X(s) \overset{ss}{=} \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2C_1 \end{array}\right].$$

Consider a square transfer function matrix $X(s)$ has the property that $X(s)$, $X^{-1}(s) \in \mathbf{RL}_\infty$ and its minimal realization is $X(s) \overset{ss}{=} [A, B, C, D]$. Since $X^{-1}(s) \in \mathbf{RL}_\infty$, then D matrix is invertible. Define the state space representation of the inverse of the transfer function matrix $X(s)$ as following:

$$X^{-1}(s) \overset{ss}{=} \left[A_{INV}, BD^{-1}, -D^{-1}C, D^{-1}\right] \tag{3.3}$$

where $A_{INV} = A - BD^{-1}C$. Let $\mathcal{T}_-(A_{INV})$ ($\mathcal{T}_+(A)$) denote the space spanned by the generalized eigenvectors of A_{INV} (A) corresponding to eigenvalues in $\mathbf{C}^-$ ($\mathbf{C}^+$). Note if $A \in \mathbf{R}^{n\times n}$, then the two subspaces $\mathcal{T}_-(A_{INV})$ and $\mathcal{T}_+(A)$ are *complementary* if their direct sum is the whole n space, i.e.,

$$\mathbf{R}^n = \mathcal{T}_-(A_{INV}) \oplus \mathcal{T}_+(A).$$

Then the following lemma gives the necessary and sufficient condition for the existence of canonical factors.

Lemma 3.1. *[BGK79, Theorem 1.5] $X(s)$ has a canonical factorization if and only if $\mathcal{T}_-(A_{INV})$ and $\mathcal{T}_+(A)$ are complementary.*

In other words, the multiplier M has a canonical factorization if and only if the invariant subspace of A corresponding to its eigenvalues with positive real part and that of A_{INV} corresponding to its eigenvalues with negative real part are complementary.

Hence given a square transfer function matrix with its minimal state space representation $X(s) = D + C(sI - A)^{-1}B$ that is bounded and invertible on $\mathbf{C}^{oe}$, we can construct the canonical factors of $X(s)$ through the following procedure[Fra87].

Step 1 Obtain full rank real matrices T_1 and T_2 such that they form the bases for the column space spanned by $\mathcal{T}_{-}(A_{INV})$ and $\mathcal{T}_{+}(A)$, respectively. Define the matrix

$$T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}.$$

and note that T is square and nonsingular because of the complementary condition.

Step 2 Perform similarity transformation such that

$$T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix}, T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$CT = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

where A_1 contains eigenvalues in $\mathbf{C}^{-}$ and A_4 has all its eigenvalues in $\mathbf{C}^{+}$.

Step 3 Define

$$X_{+}(s) \stackrel{ss}{=} [A_4, B_2, C_2, D] \tag{3.4}$$

$$X_{-}(s) \stackrel{ss}{=} \left[A_1, B_1, D^{-1}C_1, I\right] \tag{3.5}$$

Since A_1 is stable and A_4 is antistable, it follows that $X_-(s)$, $X_+^\sim(s) \in \mathbf{RH}_\infty$. Also, since

$$X_+^{-1}(s) \stackrel{ss}{=} \left[A_4 - B_2 D^{-1} C_2, B_2 D^{-1}, -D^{-1} C_2, D^{-1}\right] \tag{3.6}$$

$$X_-^{-1}(s) \stackrel{ss}{=} \left[A_1 - B_1 D^{-1} C_1, B_1, -D^{-1} C_1, I\right] \tag{3.7}$$

and $A_4 - B_2 D^{-1} C_2$ is antistable, $A_1 - B_1 D^{-1} C_1$ is stable, it then follows that $X_-^{-1}(s)$, $(X_+^{-1}(s))^\sim \in \mathbf{RH}_\infty$.

Therefore, the canonical factorization of $X(s)$ is $X(s) = X_+(s) X_-(s)$.

Next, we will introduce some terminology associated with the Hamiltonian matrix and algebraic Riccati equation.

Let A, Q, and R be real $n \times n$ matrices with Q and R symmetric. Then an *algebraic Riccati equation* is the following matrix equation:

$$A^* X + XA + XRX + Q = 0. \tag{3.8}$$

Associated with this Riccati equation is a $2n \times 2n$ matrix:

$$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix} \tag{3.9}$$

A matrix of this form is called a *Hamiltonian matrix.* The matrix H in equation (3.9) will be used to obtain the solutions to the equation (3.8). It is useful to note that the spectrum of H is symmetric about the imaginary axis.

Assume that H has no eigenvalues on the imaginary axis. Then it must have n eigenvalues in $\mathrm{Re}s < 0$ and n in $\mathrm{Re}s > 0$. Consider the n-dimensional invariant spectral subspace, $\mathcal{X}_-(H)$, stacking the basis vectors up to form a matrix, and partitioning the matrix, we get

$$\mathcal{X}_-(H) = \text{Im} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1,\ X_2 \in \mathbf{C}^{n\times n}$. ($X_1$ and X_2 can be chosen to be real matrices.) If X_1 is nonsingular or, equivalently, if the two subspaces

$$\mathcal{X}_-(H), \ \ \text{Im} \begin{bmatrix} 0 \\ I \end{bmatrix} \tag{3.10}$$

are complementary, we can set

$$X := X_2 X_1^{-1}.$$

Then X is uniquely determined by H, i.e., $H \longmapsto X$ is a function, which will be denoted Ric. We will take the domain of Ric, denoted dom(Ric), to consist of Hamiltonian matrices H with two properties:

1. H has no eigenvalues on the imaginary axis,

2. the two subspaces in equation (3.10) are complementary.

This solution will be called the stabilizing solution. Thus,

$$X = \text{Ric}(H)$$

and

$$\text{Ric} : \text{dom}(\text{Ric}) \subset \mathbf{R}^{2n\times 2n} \mapsto \mathbf{R}^{n\times n}.$$

Therefore, for any Hamiltonian matrix, $H \in \mathbf{R}^{2n\times 2n}$, if there exists $X = X^T \in \mathbf{R}^{n\times n}$ such that

$$H \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} \Lambda, \ \operatorname{spec}(\Lambda) = \operatorname{spec}(H) \cap \mathbf{C}^-, \tag{3.11}$$

then write $H \in \operatorname{dom}(\operatorname{Ric})$, and $X = \operatorname{Ric}(A)$, see, e.g., [DGKF89, ZDG96].

3.4 Stable Factors of Quadratic Constraints

In this section, the quadratic constraints in the IQC framework can be shown to be unimodularly congruent to an indefinite constant matrix. In other words, there exists a stable and minimum phase factor for the factorization of the quadratic constraints.

Consider a square transfer function matrix $\Pi(s)$ with state space representation of order n as follows:

$$\Pi(s) = D + C(sI - A)^{-1}B. \tag{3.12}$$

Given that $\Pi(s), \Pi^{-1}(s) \in \mathbf{RL}_\infty$ and $\Pi(s) = \Pi^\sim(s)$, clearly $\Pi(s)$ has no eigenvalues on $\mathbf{C}^{oe}$. Let n_1 denotes the number of stable eigenvalues of A. Since $\Pi(s) = \Pi^\sim(s)$, there are same number of unstable eigenvalues of A. It then holds that there exist state space similarity transformations which enable us to write

$$\begin{aligned} \Pi(s) &\overset{ss}{=} [A, B, C, D] \\ &= \left[\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 C_2 \end{bmatrix}, D \right] \\ &= D + [A_1, B_1, C_1, 0] + [A_2, B_2, C_2, 0] \end{aligned}$$

with spec(A_1)$\subset$ $\mathbf{C}^-$ and spec(A_2)$\subset$ $\mathbf{C}^+$ where

$$A_2 = -A_1^T, B_2 = -C_1^T, C_2 = B_1^T.$$

Therefore, $\Pi(s)$ can be factored as

$$\begin{aligned} \Pi(s) &= D + \Pi_1(s) + \Pi_1^{\sim}(s) \\ &= D + [A_1, B_1, C_1, 0] + \left[-A_1^T, -C_1^T, B_1^T, 0\right] \end{aligned}$$

For convenience, we henceforth redefine A, B, and C as follows,

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & -A_1^T \end{bmatrix}, B = \begin{bmatrix} B_1 \\ -C_1^T \end{bmatrix}, C = \begin{bmatrix} C_1 & B_1^T \end{bmatrix}. \tag{3.13}$$

We will now show that a stable factorization of $\Pi(s)$ can always be constructed.

Lemma 3.2. *Consider square matrix transfer function $\Pi(s) = \Pi^{\sim}(s)$, bounded and invertible on $\mathbf{C}^{oe}$ defined by (3.13), with spec(A_1)$\subset$ $\mathbf{C}^-$. It holds that*

1. *A_{INV} $\in$dom(Ric), where $A_{INV} = A - BD^{-1}C$.*

2. $$\Pi(s) = \Pi_-^{\sim}(s) D \Pi_-(s). \tag{3.14}$$

   where

$$\begin{aligned} \Pi_-(s) &= I + C_-(sI - A_1)^{-1}B_-, && (3.15) \\ C_- &= D^{-1}(C_1 + B_1^T P), && (3.16) \\ B_- &= B_1, && (3.17) \\ P &= Ric(A_{INV}). && (3.18) \end{aligned}$$

Proof. Since $\Pi^{-1}(s) \in \mathbf{RL}_\infty$, we have the existence of D^{-1}. Since the state space representation of $\Pi(s)$ has the form of equation (3.13), therefore the inverse of $\Pi(s)$ has the following state space representation:

$$\Pi^{-1}(s) = \left[A_{INV}, BD^{-1}, -D^{-1}C, D^{-1}\right]$$

where

$$\begin{aligned} A_{INV} &= A - BD^{-1}C \\ &= \begin{bmatrix} A_1 - B_1 D^{-1} C_1 & -B_1 D^{-1} B_1^T \\ C_1^T D^{-1} C_1 & -(A_1 - B_1 D^{-1} C_1)^T \end{bmatrix}. \end{aligned} \tag{3.19}$$

Note that $\Pi(s)$ has pole and zero symmetry about the imaginary axis. Therefore, the eigenvalues of A_{INV} are the poles of $\Pi^{-1}(s)$ or the zeros of $\Pi(s)$. Comparison the structure of A_{INV} and the definition of Hamiltonian matrix show that A_{INV} is a Hamiltonian matrix and has no eigenvalues on the imaginary axis (due to its pole and zero symmetry). By [DGKF89], for any Hamiltonian matrix, if there exists $P = P^T$ such that

$$A_{INV} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} \Lambda.$$

where $\mathrm{spec}(\Lambda) = \mathrm{spec}(A_{INV}) \cap \mathbf{C}^-$, then

$$A_{INV} \in \mathrm{dom}(\mathrm{Ric}) \text{ and } P = \mathrm{Ric}(A_{INV}),$$

and we have proved item 1.

Given item 1, then item 2 follows by canonical factorization. From the A matrix we see that $\mathcal{T}_+(A)=\text{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$. From item 1, A_{INV} has no eigenvalues on the imaginary axis and $A_{INV} \in \text{dom}(Ric)$ which means that there exists a matrix $P = P^T$ such that $\mathcal{T}_-(A_{INV})=\text{Im}\begin{bmatrix} I \\ P \end{bmatrix}$. Due to the pole and zero symmetry of $\Pi(s)$, the two subspaces $\mathcal{T}_+(A)$ and $\mathcal{T}_-(A_{INV})$ are complementary. So by Lemma 3.1, $\Pi(s)$ has a canonical factorization. Now defining the nonsingular matrix

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix},$$

we get

$$T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ -PA_1 + A_1^T P & -A_1^T \end{bmatrix},$$

$$T^{-1}B = \begin{bmatrix} B_1 \\ -(C_1^T + PB_1) \end{bmatrix},$$

$$CT = \begin{bmatrix} C_1 + B_1^T P & B_1^T \end{bmatrix}.$$

By analogy with (3.4) and (3.5) we have$\Pi(s) = \Pi_+(s) D \Pi_-(s)$ where

$$\Pi_+(s) \overset{ss}{=} \left[-A_1^T, -(C_1^T + PB_1)D^{-1}, B_1^T, I\right] \tag{3.20}$$

$$\begin{aligned} \Pi_-(s) &\overset{ss}{=} [A_1, B_-, C_-, I], \\ &= \left[A_1, B_1, D^{-1}(C_1 + B_1^T P), I\right], \end{aligned} \tag{3.21}$$

with spec(A_1)$\subset \mathbf{C}^-$. Note that $\Pi_+(s) = \Pi_-^\sim(s)$, therefore we can rewrite $\Pi(s)$ as

$$\Pi(s) = \Pi_-^\sim(s) D \Pi_-(s) \tag{3.22}$$

where

$$\begin{aligned} \Pi_-(s) &= I + C_-(sI - A_1)^{-1} B_- \\ C_- &= D^{-1}(C_1 + B_1^T P) \\ B_- &= B_1. \end{aligned}$$

□

Remark 3.2. *The two simplest examples of* $\Pi(s)$ *are*

$$\Pi_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad , \textit{ and } \quad \Pi_2 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

We see that Π_1 *and* Π_2 *represent the small gain theorem and the passivity theorem, respectively.*

Remark 3.3. *The two weighting functions* Π_1 *and* Π_2 *are congruent to each other.*

$$\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \right) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \right).$$

Theorem 3.1. *For the square transfer function matrix* $\Pi(s) = \Pi^\sim(s)$*, bounded and invertible on* $\mathbf{C}^{oe}$ *that satisfies (3.14), then* $\Pi(s)$ *is unimodularly congruent to a signature matrix* $J = diag(I, -I)$ *if and only if there exists a nonsingular matrix* T *such that* $D = T^T J T$.

Proof. Since the square transfer function matrix $\Pi(s) = \Pi^\sim(-s)$, bounded and invertible on $\mathbf{C}^{oe}$ that satisfies (3.14), we have $\Pi(s) = \Pi_-^\sim(s) D \Pi_-(s)$. The D matrix is nonsingular and symmetric. If there exists a nonsingular matrix T such that $D = T^T JT$. Substituting D into $\Pi(s)$, we can get

$$\begin{aligned} \Pi(s) &= \Pi_-^{\sim}(s) T^T J T \Pi_-(s) \\ &= (T\Pi_-(s))^{\sim} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} (T\Pi_-(s)) \\ &= S^{\sim}(s) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S(s) \end{aligned} \tag{3.23}$$

where $S(s) = T\Pi_-(s)$. □

Hence, by Theorem 3.1 the square transfer function matrix $\Pi(s) = \Pi^{\sim}(s)$ can always be factored as follows:

$$\Pi(s) = S^{\sim}(s) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S(s)$$

where $S(s)$ is a stable, minimum phase transfer function matrix. The only requirement for $\Pi(s)$ is its invertibility, i.e., the D matrix of its state space representation is nonsingular. Therefore, by congruent transformation of D matrix we can show that the invertible, bounded square transfer function matrix $\Pi(s)$ is unimodularly congruent to an indefinite constant matrix $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. In fact, the indefinite J matrix is congruent to another form of indefinite constant matrix.

Remark 3.4. *By Remark 3.3, the weighting function* $\Pi(s)$ *could be transformed into the positivity form as follows*

$$\begin{aligned}
\Pi(s) &= S^{\sim}(s) \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} S(s) \\
&= S^{\sim}(s) \left(\frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \right) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \left(\frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \right) S(s) \\
&= \hat{S}^{\sim}(s) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \hat{S}(s)
\end{aligned}$$

where $\hat{S}(s) = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} S(s)$.

Remark 3.5. *Based on the derivation of canonical factorization from [Fra87, BGK79], Hassibi* et al. *[HSK99] also presented the* J*-canonical factorization of rational functions. Indeed, this* J*-canonical factorization of rational functions is similar to our unimodularly congruent results in this chapter.*

3.5 Summary

Working from the quadratic separating functionals used for stability and robustness analysis of the feedback systems, we show that a stable, minimum phase state space factors always exists for the quadratic constraints in the topological framework and IQC framework for robust analysis. As a consequence, the existence of stable and stability invertible sector transformation form of spectral factors is proved.

In the next chapter, we will apply the results of this chapter to show that the generalized multiplier IQCs are unimodularly congruent to a positivity form and have a special diagonal form for the stable, minimum phase factors.

Chapter 4

Factorization of Generalized Multiplier IQCs

4.1 Introduction

The use of multipliers in stability analysis with the small gain theorem or the passivity theorem can generally reduce conservatism of the analysis extensively. In general, the multiplier M and its inverse are assumed to be bounded but not necessarily causal. However, the passivity theorem requires causal operators in the feedback interconnection and it can therefore not be applied to the system if M or M^{-1} is noncausal. In this case it is required that there exists a canonical factorization $M = M_+ M_-$, where M_-, M_+^* and their inverses are causal and bounded. If such a factorization exists, the stability conditions can be stated in terms of IQCs involving the multiplier M. Therefore, the passivity theorem with multiplier can be reformulated with IQC defined by $\Pi(j\omega) = \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & 0 \end{bmatrix}$.

In this chapter, we will show that the generalized multiplier IQCs always exists a stable factorization in a specific form. Based on the results of canonical factorization for generalized positive real transfer function [Goh96], we can show that the generalized multiplier IQCs is unimodularly congruent to an indefinite constant matrix with a stable and minimum phase diagonal factor.

4.2 Generalized Multipliers in IQC Framework

In the early 60s, the principle of partitioning the open loop into two positive operators was introduced in the stability analysis of time-varying nonlinear feedback system. See [Pop62, Yak67, Zam66, Wil72, DV75, Saf80] and the reference therein. The two most famous fundamental results in stability theory are small gain theorem and passivity theorem. However, both theorems deal with general classes of operators. Hence, in situations where more information about the operators is available, these theorems possibly yield conservative results. This difficulty can in theory be partially overcome by scalings in the small gain framework, and by multiplier theory in the passivity approach. Therefore, the use of multipliers in stability analysis with the passivity theorem can generally reduce conservatism of the analysis extensively. In this section, we will discuss the classical multiplier theory and relate it to the IQC approach for stability analysis.

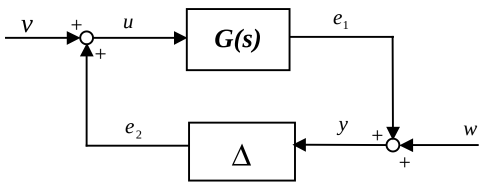

Figure 4.1: System interconnection in the classical input-output theory.

Assume that we want to study the stability of system $\mathcal{S}_1$ in Figure 4.1 with the passivity theorem. We will first introduce the passivity theorem.

Theorem 4.1 (Passivity Theorem). *Assume that the feedback interconnection of G and Δ in Figure 4.1 is well-posed and the following conditions hold*

$$\begin{aligned} \langle u_\tau, Gu_\tau \rangle &\leq -\epsilon \left\| u_\tau \right\|^2 \\ \langle u_\tau, \Delta u_\tau \rangle &\geq 0 \end{aligned} \tag{4.1}$$

for all $u \in \boldsymbol{L}^m_{2e,[0,\infty)}$. The system is then stable.

Proof. See [DV75] for a full proof. □

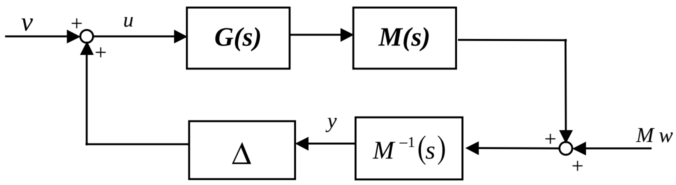

Figure 4.2: System interconnection with multiplier inserted in the loop.

In the classical input-output theory a generalized invertible multiplier M is inserted in the loop resulting in system $\mathcal{S}_2$ in Figure 4.2. This generalized multiplier is also assumed to be a bounded linear operator. The multiplier M and its inverse are assumed to be bounded but not necessarily causal.

However, the passivity theorem requires causal operators in the feedback interconnection and it can therefore not applied to system $\mathcal{S}_2$ if M and M^{-1} is noncausal. In this case it is required that M can be factored into $M = M_- M_+$, where M_-, M_-^{-1}, M_+^*, $\left(M_+^*\right)^{-1}$ are causal and bounded. If such factorization exists, we can use the following lemma from [ZF68].

Lemma 4.1. *The following are equivalent:*

1. *For some* $\epsilon > 0$

$$\begin{aligned} \langle v, MGv \rangle &\leq -\epsilon \|v\|^2 \\ \langle v, M^* \Delta(v) \rangle &\geq 0 \end{aligned} \tag{4.2}$$

   for all $v \in \boldsymbol{L}^m_{2,[0,\infty)}$.

2. *For some* $\epsilon > 0$

$$\begin{aligned} \left\langle u_\tau, M_- G (M_+^*)^{-1} u_\tau \right\rangle &\leq -\epsilon \|u_\tau\|^2 \\ \left\langle u_\tau, M_+^* \Delta(M_-^{-1} u_\tau) \right\rangle &\geq 0 \end{aligned} \tag{4.3}$$

   for all $u \in \boldsymbol{L}^m_{2e,[0,\infty)}$ *and for all* $\tau \geq 0$.

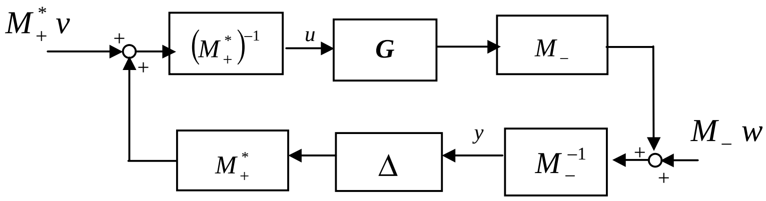

Figure 4.3: System interconnection with multiplier factorization.

Consider now system $\mathcal{S}_3$ in Figure 4.3. By Lemma 4.1, the stability and well-posedness of system $\mathcal{S}_1$ and system $\mathcal{S}_3$ are equivalent conditions. This follows since all the multipliers in system $\mathcal{S}_3$ are bounded and causal. Therefore, by applying the passivity theorem and IQC framework to system $\mathcal{S}_3$, we can have the following multiplier IQCs theorem.

Theorem 4.2 (Multiplier IQCs Theorem). *Assume that*

1. *the feedback interconnection of* G *and* Δ *is well-posed,*

2. Δ satisfies the IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} \tag{4.4}$$

3. M can be factored into $M = M_- M_+$, where M_-, M_-^{-1}, M_+^*, $\left(M_+^*\right)^{-1}$ are causal and bounded

4. there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \ \forall \omega \in \boldsymbol{R}. \tag{4.5}$$

Then the interconnection of G and Δ is stable.

Therefore, the passivity theorem with the generalized multiplier can be reformulated with IQC defined by (4.4). On the other hand, the generalized positive real (GPR) multiplier approach for robust analysis, e.g., [Bal95, SC93, LSC94, HHHB94], based on the same earlier work of [ZF68] also yields similar stability conditions. Hence, the generalized multipliers used in the IQC defined by (4.4) are actually the generalized positive real multipliers.

In the following sections, based on the canonical factorization of the generalized positive real multiplier, we will show that the generalized multiplier IQC can also be unimodularly congruent to a positivity form.

4.3 Canonical Factorization of Generalized Multipliers

In this section, we will show that a minimal state space canonical factorization always exists for every generalized positive real (GPR) transfer function matrix, i.e., given the state space matrices of GPR $M(s)$, with n_1 (n_2) poles in $\mathbf{C}^-$ ($\mathbf{C}^+$), it is always possible to construct the state space realization of $M_-(s)$ and $M_+(s)$ of order n_1 and n_2 respectively, with $M_-(s)$ and $M_+(-s)$ bounded and invertible over $\overline{\mathbf{C}^+}$, such that .

In order to show that the generalized multiplier IQC of (4.4) always exists a stable factorization in a specific diagonal form. We first need to show that given any square generalized positive real transfer function matrix $M(s)$, there always exists a canonical factorization $M(s) = M_+(s)M_-(s)$, where M_-, M_-^{-1}, M_+^*, $(M_+^*)^{-1}$ are causal and bounded.

Consider an r input r output real rational transfer function matrix $M(s)$, bounded, invertible and positive real over $\mathbf{C}^{oe}$, the extended $j\omega$-axis, i.e., there exists $\epsilon > 0$ such that

$$\epsilon^{-1} I_r \geq \text{herm}\{M(s)\} \geq \epsilon I_r, \ \forall s \in \mathbf{C}^{oe} \tag{4.6}$$

where, for any real rational transfer matrix, $X(s)$, $\text{herm}\{X(s)\} := X(s) + X^{\sim}(s)$ and $X^{\sim}(s) := X^T(-s)$.

Definition 4.1. *If a square real rational transfer function matrix $M(s)$ satisfies (4.6) for some $\epsilon > 0$, then $M(s)$ is a (strict) generalized positive real (GPR) transfer function matrix.*

Next, we define some terminology associated with the theory of Riccati equation. See also [BGKD80]. Consider the matrix $X \in \mathbf{R}^{p\times p}$, the inertia of X is given by $in(X) = (a, b, c)$, where a, b, c are the number of elements of $\text{spec}(X)$ in $\mathbf{C}^+$,

$\mathbf{C}^-$ and $\mathbf{C}^{oe}$ respectively, counting multiplicities. Suppose X has inertia $in(X) = (p_1, p_2, 0)$ for two fixed integers p_1, p_2 and $p = p_1 + p_2$. Partition X as follows: $X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$ with $X_{i,j} \in \mathbf{R}^{p_i \times p_j}$. Then there always exists $Q_1 \in \mathbf{R}^{p_1 \times p_1}$, $Q_2 \in \mathbf{R}^{p_2 \times p_1}$ such that

$$X \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \Lambda, \tag{4.7}$$

where $\Lambda \in \left\{\Lambda \in \mathbf{R}^{p_1 \times p_1} : \text{spec}(\Lambda) = \text{spec}(X) \cap \mathbf{C}^-\right\}$. If additionally Q_1 is invertible, and defining $Q := Q_2 Q_1^{-1}$, then one has

$$X \begin{bmatrix} I \\ Q \end{bmatrix} = \begin{bmatrix} I \\ Q \end{bmatrix} \Lambda, \tag{4.8}$$

where $\Lambda \in \left\{\Lambda \in \mathbf{R}^{p_1 \times p_1} : \text{spec}(\Lambda) = \text{spec}(X) \cap \mathbf{C}^-\right\}$, and one may then associate with X the following quadratic matrix equation

$$\begin{bmatrix} Q & -I_{p_2} \end{bmatrix} X \begin{bmatrix} I_{p_1} \\ Q \end{bmatrix} = QX_{11} - X_{22}Q + QX_{12}Q - X_{21} = 0 \tag{4.9}$$

which we term the generalized algebraic Riccati equation (GARE), since (4.9) is a more general form of the standard algebraic Riccati equation of, e.g., [DGKF89, Fra87, LR91].

Definition 4.2. *For any square matrix X, write $X \in dom(gRic)$ if,*

1. *X has no $\mathbf{C}^{oe}$ eigenvalues, and*

2. given that X has p_1 eigenvalues in $\boldsymbol{C}^-$, there exists $Q_1 \in \boldsymbol{R}^{p_1 \times p_1}$, which is invertible, $Q_2 \in \boldsymbol{R}^{p_2 \times p_1}$ and $\Lambda \in \boldsymbol{R}^{p_1 \times p_1}$ such that (4.7) holds. Write then

$$Q_2 Q_1^{-1} := Q = gRic(X).$$

We call Q a stabilizing solution to the generalized Riccati equation (4.9).

Based on the stabilizing solutions of GARE from this section and the definition of canonical factorization and the existence conditions from Chapter 3, [Goh96] has derived the canonical factorization of the GPR transfer function matrix.

Consider the transfer function matrix $M(s)$ with state space representation of order n as follows:

$$M(s) = D_m + C_m(sI_n - A_m)^{-1}B_m. \tag{4.10}$$

Given that $M(s)$ is such that (4.6) holds for some $\epsilon > 0$, clearly A_m has no eigenvalues on $\mathbf{C}^{oe}$. For convenience, we redefine A_m, B_m, C_m and D_m as follows:

$$A_m := \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},\ B_m := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ C_m := \begin{bmatrix} C_1 & C_2 \end{bmatrix} \text{and } D_m := D.$$

Since $M(s)$ is such that (4.6) holds for $s = \infty$, then D_m is invertible. Define

$$\begin{aligned} A^\times &:= A_m - B_m D_m^{-1} C_m \\ &= \begin{bmatrix} A_1 - B_1 D^{-1} C_1 & -B_1 D^{-1} C_2 \\ -B_2 D^{-1} C_1 & A_2 - B_2 D^{-1} C_2 \end{bmatrix}. \end{aligned} \tag{4.11}$$

Clearly, the zeros of $M(s)$ are the eigenvalues of $A^\times$. Note that if $M(s)$ is GPR, then $M(s)$ has an equal number of poles and zeros in the open right and left half complex planes.

The following theorem will show that a state space canonical factorization for a GPR $M(s)$ can always be constructed.

Theorem 4.3 (GPR Canonical Factorization). *Consider $M(s)$ bounded and invertible on $\boldsymbol{C}^{oe}$ defined by (4.10), with spec$(A_1) \subset \boldsymbol{C}^-$ and spec$(A_2) \subset \boldsymbol{C}^+$. If $M(s)$ is GPR, it then holds that*

1. *Given $A^\times$ of (4.11), then $A^\times \in$ dom(gRic).*
2. *There exist $r \times r$ $M_-(s)$ and $M_+(s)$, with $M_-(s)$ and $M_+(-s)$ bounded and invertible on $\overline{\boldsymbol{C}^+}$, such that $M(s) = M_+(s)M_-(s)$. Given $P = gRic(A^\times)$ from item (1), one possible realization pair for $M_-(s)$ and $M_+(s)$ is*

$$M_-(s) \;:=\; I_r + C_{M_1}(sI_{n_1} - A_1)^{-1}B_1, \tag{4.12}$$

$$M_+(s) \;:=\; D + C_2(sI_{n_2} - A_2)^{-1}B_{M_2}, \tag{4.13}$$

where

$$C_{M_1} \;:=\; D^{-1}\left[C_1 + C_2P\right], \tag{4.14}$$

$$B_{M_2} \;:=\; B_2 - PB_1. \tag{4.15}$$

Proof. See [Goh96, Theorem 1] for detail proof. □

In summary, every generalized positive real multiplier $M(s)$ with n_1 (n_2) poles in $\mathbf{C}^-$ ($\mathbf{C}^+$) can always be possible to construct the state space realization of $M_-(s)$ and $M_+(s)$, with $M_-(s)$, $M_+^{\sim}(s)$ bounded and invertible over $\overline{\mathbf{C}^+}$, the closed right half complex plane, such that $M(s) = M_+(s)M_-(s)$. Note that the formulae are given in terms of a stabilizing solution to a generalized algebraic Riccati equation which is shown to always exist if $M(s)$ is generalized positive real.

4.4 Stable Factors of Generalized Multiplier IQCs

We have shown that given any square generalized positive real transfer function matrix $M(s)$, there always exist a canonical factorization $M(s) = M_+ M_-$, where

$$M_-(s) = I + C_{M_1}(sI - A_1)^{-1}B_1,$$
$$M_+(s) = D + C_2(sI - A_2)^{-1}B_{M_2}$$

with $\mathrm{spec}(A_1) \subset \mathbf{C}^-$ and $\mathrm{spec}(A_2) \subset \mathbf{C}^+$. Note that the state space formulae for the canonical factors are presented in terms of a stabilizing solution to a generalized Riccati equation which is shown to always exist if $M(s)$ is generalized positive real.

Thus, based on the canonical factorization of the generalized positive real multipliers, we can always get the stable and minimum factorization of the corresponding multiplier IQCs. In the following theorem, the generalized multiplier IQCs are shown to be unimodularly congruent to an indefinite constant matrix and have a stable and minimum phase diagonal factor.

Theorem 4.4 (Stable Factorization of Multiplier IQC). *Given any square bounded, invertible and positive real transfer function matrix $M(s)$, with canonical factorization $M(s) = M_+ M_-$, then the multiplier IQC*

$$\Pi(s) = \begin{bmatrix} 0 & M^{\sim}(s) \\ M(s) & 0 \end{bmatrix}$$

can always have a stable factorization as follows:

$$\Pi(s) = \begin{bmatrix} M_-^{\sim} & 0 \\ 0 & M_+ \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M_- & 0 \\ 0 & M_+^{\sim} \end{bmatrix}. \tag{4.16}$$

Proof. Since $M(s)$ has a canonical factorization $M(s) = M_+ M_-$ with $M_-(s)$, $M_+^\sim(s)$ and their inverses are causal and bounded, the generalized multiplier IQC can be written as

$$\Pi(s) = \begin{bmatrix} 0 & M^\sim(s \\ M(s) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ M(s) & 0 \end{bmatrix} + \begin{bmatrix} 0 & M^\sim(s) \\ 0 & 0 \end{bmatrix}.$$

Since those two parts of $\Pi(s)$ are conjugate transpose to each other, it suffices to take one of them and to show that

$$\begin{bmatrix} 0 & 0 \\ M(s) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & M_+ \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M_- & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.17}$$

Note that the state space representation of the generalized multiplier can be shown as following:

$$\begin{aligned} M(s) &= M_+(s)M_-(s) \\ &\overset{ss}{=} \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_{M_2}C_{M_1} & A_2 & B_{M_2} \\ \hline DC_{M_1} & C_2 & D \end{array}\right] \\ &\overset{ss}{=} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & D_m \end{array}\right]. \end{aligned}$$

Then the state space representation of the left-hand side of (4.17) can be shown as following:

$$\begin{bmatrix} 0 & 0 \\ M(s) & 0 \end{bmatrix} \stackrel{ss}{=} \left[\begin{array}{c|cc} A_m & B_m & 0 \\ \hline 0 & 0 & 0 \\ C_m & D_m & 0 \end{array}\right]$$

$$\stackrel{ss}{=} \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ B_{M_2}C_{M_1} & A_2 & B_{M_2} & 0 \\ \hline 0 & 0 & 0 & 0 \\ DC_{M_1} & C_2 & D & 0 \end{array}\right]. \tag{4.18}$$

On the other hand, we can show that the state space representation of the right-hand side of (4.17) is the following:

$$\begin{bmatrix} 0 & 0 \\ 0 & M_+ \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M_- & 0 \\ 0 & 0 \end{bmatrix}$$

$$\stackrel{ss}{=} \left[\begin{array}{c|cc} A_2 & 0 & B_{M_2} \\ \hline 0 & 0 & 0 \\ C_2 & 0 & D \end{array}\right] \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & I \\ 0 & I & 0 \end{array}\right] \left[\begin{array}{c|cc} A_1 & B_1 & 0 \\ \hline C_{M_1} & I & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\stackrel{ss}{=} \left[\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ B_{M_2}C_{M_1} & A_2 & B_{M_2} & 0 \\ \hline 0 & 0 & 0 & 0 \\ DC_{M_1} & C_2 & D & 0 \end{array}\right]. \tag{4.19}$$

Therefore, by comparing the state space form of (4.18) and (4.19), it is trivial to show that

$$\begin{bmatrix} 0 & M^{\sim}(s) \\ M(s) & 0 \end{bmatrix} = \begin{bmatrix} M_{-}^{\sim} & 0 \\ 0 & M_{+} \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} M_{-} & 0 \\ 0 & M_{+}^{\sim} \end{bmatrix}.$$

Note that the factor $\begin{bmatrix} M_{-} & 0 \\ 0 & M_{+}^{\sim} \end{bmatrix}$ is stable and minimum phase due to the fact that the GPR multiplier has canonical factorization.

□

Hence, the generalized multiplier IQCs is unimodularly congruent to an indefinite constant matrix, i.e., $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, and the stable, minimum phase factor has a diagonal form, i.e., $\begin{bmatrix} M_{-} & 0 \\ 0 & M_{+}^{\sim} \end{bmatrix}$.

4.5 Summary

In this chapter, we first showed existence of the canonical factorization of the generalized positive real (GPR) multipliers. Based on the canonical factorization results, we showed that the quadratic constraints with the generalized positive real multipliers in the IQC framework can be unimodularly congruent to an indefinite constant matrix and the stable, minimum phase factor has a diagonal form.

Chapter 5

Unstably-Weighted Robust Control Synthesis

5.1 Introduction

This chapter is mainly motivated by the works of "M-K iteration". In the process of "M-K iteration" the usual diagonal scalings are replaced with unstable weighting matrix $M(s)$. The solution of [SC93] for robust controller synthesis involves a preliminary multiplier factorization and bilinear transformation, which reduce the problem to a standard $\mathcal{H}_\infty$ control problem (e.g., [MH77, Saf83, DGKF89, GLD+91, Gah92, IS94]). However, while working on the "M-K iteration" there has been until now an intermediate step that requires to compute the factorization of the unstable weighting matrix transfer function $M(s) = (M_2^T(s))^{-1}M_1(s)$ where $M_1(s)$, $M_2(s)$ and their inverses are stable and minimum phase. This factorization step certainly introduces computation complexity and potential numerical problems.

It is shown in this chapter that the synthesis of feedback controller can be consider as solving the unstably-weighted closed-loop system $M(s)T(s)$ is generalized strongly positive real and the closed-loop system $T(s)$ is stable. This approach will enable direct design of a robust controller without having to first compute equivalent stable weighting function $M(s)$ in the positive real synthesis framework ([WWS94, SKS94, THS95]).

5.2 μ/k_m-Synthesis via M-K Iteration

In the "$D-K$ iteration" approach for μ/k_m-synthesis, we can see that the improvement of robustness during μ/k_m-synthesis process strongly depends on the quality of the curve fitting; so this step in the "$D-K$ iteration" approach has been seriously criticized as the weak link in μ/k_m-synthesis. To alleviate those difficulties, Safonov *et al.* [SC93, SLC93] have developed methods to study the so-called "mixed μ/k_m-synthesis problem" which resulting the "$M-K$ iteration".

In order to bypass the awkward curve fitting step in μ/k_m-synthesis, Safonov and coworkers proposed a multiplier approach to compute suitable scalings [CS92, SL93, SC93, SLC93, LSC94]. In this approach, the $\mathcal{H}_\infty$ minimization problem is transformed to an equivalent positive real problem in which a convex parameterization of the scalings in the form of a multiplier is possible. In this form, scaling can be computed effectively and without curve fitting. Other approaches using a similar transformation include [BHPD94, HH93].

The multiplier framework of the "$M-K$ Iteration" operates on the positive-real bilinear sector transformed system of Figure 5.1 where the uncertainty $\Delta(s) \in \mathcal{D}_\mathcal{K}$ and $T(s)$ is any stable rational transfer function matrix such that $\|T(s)\|_\infty < 1$. The sector bounded uncertainty structure $\mathcal{D}_\mathcal{K}$ can be defined as,

$$\begin{aligned}\tilde{\mathcal{D}_\mathcal{K}} \ := \ & \Big\{\tilde{\Delta(s)} = \operatorname{diag}\Big(\tilde{\delta}_1^r I_{k_i}, \cdots, \tilde{\delta}_{m_r}^r I_{k_{m_r}}, \tilde{\delta}_1^c(s) I_{k_{m_r+1}}, \cdots, \tilde{\delta}_{m_c}^c(s) I_{k_{m_r+m_c}}, \\ & \tilde{\Delta}_1^c(s), \cdots, \tilde{\Delta}_{m_C}^c(s)\Big) : \\ & \delta_i^r \in \mathbf{R}, \delta_i^r \leq 0, \forall i = 1, \cdots, m_r;\end{aligned}$$

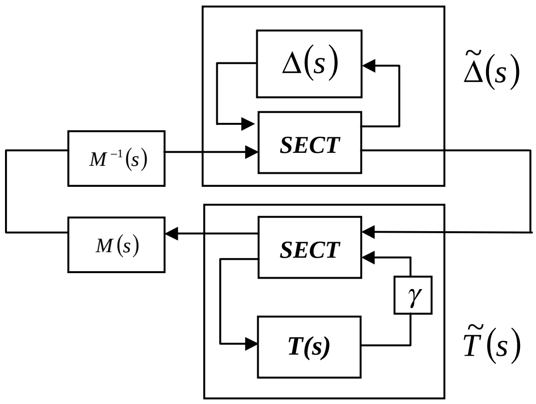

Figure 5.1: Multiplier $M-K$ Iteration without Factorization

$$\left.\begin{aligned} &\delta_i^c(s) \in \mathbf{RH}_\infty, \mathrm{herm}(\delta_i^c(s)) \leq 0, \forall i = 1, \cdots, m_c; \\ &\Delta_i^c(s) \in \mathbf{RH}_\infty^{k_{m_r+m_c+i} \times k_{m_r+m_c+i}}, \mathrm{herm}(\Delta_i^c(s)) \leq 0, \forall i = 1, \cdots, m_C \end{aligned}\right\} \tag{5.1}$$

A corresponding multiplier class may be defined, see [SC93, BHPD94, LSC94, HH93].

$$\mathcal{M} = \{M(s) = D(s) + jG(s) : D(s) \in \mathcal{M}_D, G(s) \in \mathcal{M}_G\}. \tag{5.2}$$

Hence, the multiplier $M(s) \in \mathcal{M}$ is generalized strongly positive real, i.e., there exists $\epsilon > 0$ such that

$$\frac{I}{\epsilon} \geq \mathrm{herm}\left(M(s)\right) \geq \epsilon I, \forall s \in \mathbf{C}^{oe}. \tag{5.3}$$

Moreover, $M(s)$ also has its inverse $M^{-1}(s) \in \mathcal{M}$. It commutes with $\tilde{\Delta}(s) \in \tilde{\mathcal{D}}_{\mathcal{K}}$ and also preserves positivity. Note that $M(s) \in \mathcal{M}$ can always be factored as $M(s) = M_2^{-T}(-s)M_1(s)$ [DV75, Fra87] where $M_1(s)$ and $M_2(s)$ are stable and minimum phase. Therefore, an alternative $M-K$ iteration with factorization phase may refer to Figure 5.2.

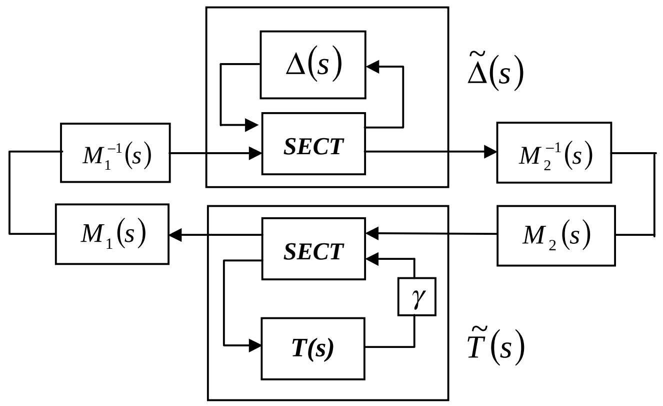

Figure 5.2: Fixed-order Multiplier $M - K$ Iteration with Factorization

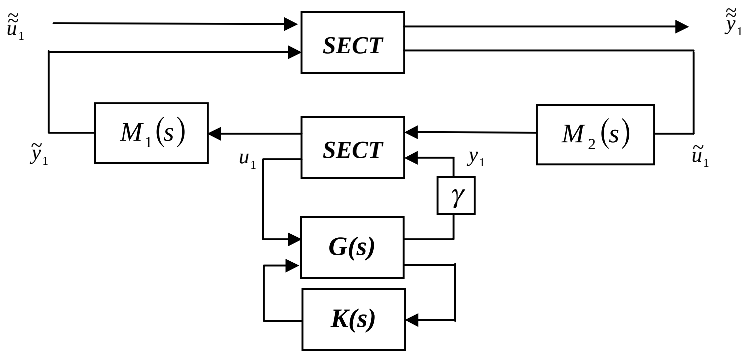

Figure 5.3: Fixed-order Multiplier μ/k_m-synthesis

With reference to Figure 5.3, the multiplier μ/k_m-synthesis goes as flows:

$$\begin{gathered}\max\gamma \\ \text{s.t.} K(s) \text{be stabilizing} \\ M_2^{-T}(-s)M_1(s) \in \mathcal{M} \\ \left\| T_{\tilde{y_1}\tilde{\tilde{u}}_1}(s) \right\|_\infty < 1.\end{gathered}$$

Following is the fixed-order multiplier $M - K$ iteration of μ/k_m-synthesis in [SC93, SLC93].

Step 1 Initialize by solving the conventional $\mathcal{H}_\infty$ optimal control problem of finding a stabilizing $\mathcal{H}_\infty$ controller $K(s)$ which maximizes the values of γ for which $\|T_{y_1u_1}(s)\|_\infty < 1$. Set $M_{best}(s) = M_1(s) = M_2(s) = I$, $K_{best}(s) = K(s)$, $\gamma_{best} = \gamma_{oldbest} = \gamma$.

Step 2 Iteratively increase γ and solve the convex optimization problem of computing a fixed-order multiplier $M(s)$ such that $M(s) \in \mathcal{M}$ which maximizes

$$\rho_M := \min_{\forall\omega, \|x\|=1} x^* \left(\text{herm}\left\{M(j\omega)T_{\tilde{y_1}\tilde{u_1}}(j\omega)\right\}\right) x. \tag{5.4}$$

If $\rho_M > 0$ set $\gamma_{best} = \gamma$, $M_{best}(s) = M(s)$ and repeat Step 2; otherwise, continue to Step 3.

Step 3 Compute the factorization $M_{best}(s) = \left(M_2^{-T}(-s)\right)^{-1} M_1(s)$ and augment the plant as shown in Figure 5.3.

Step 4 Iteratively increase γ and solve the $\mathcal{H}_\infty$ optimal control problem finding a stabilizing controller $K(s)$ which maximizes the cost

$$\rho_K := \min_{\text{stabilizing} K(s)} \left\| T_{\tilde{y_1}\tilde{\tilde{u}}_1}(s) \right\|_\infty . \tag{5.5}$$

If $\rho_K < 1$ set $K_{best}(s) = K(s)$, $\gamma_{best} = \gamma$ and repeat Step 4; otherwise go to Step 5.

Step 5 If $\gamma_{oldbest} < \gamma_{best}$, set $\gamma_{oldbest} = \gamma_{best}$ and go to Step 2; otherwise stop.

In summary, the benifits of $M - K$ Iteration are the following:

- Compute the multivariable stability margin with respect to mixed and repeated real/complex uncertainties.
- It can avoid the time consuming frequency sweep of the conventional method [Doy82].
- Unlike the approach in [BHPD94] requires expressing the multiplier $M(s)$ in terms of a basis expansion which is not clear and can introduce conservatism. The $M - K$ iteration [SC93, LSC94] does not suffer from these weakness but always produces and unstable multiplier.

Despite of those benefits, however, there still involved a multiplier factorization step which may sometimes bring computational complexity to $M - K$ iteration.

Remark 5.1. *In [FTD91] the so-called G-scales matrices are introduced in addition to the D-scales to refine the upper bound on the structure singular value μ in case of mixed real/complex uncertainties. More recent results extend the analysis to the synthesis problem by using a D, $G - K$ iteration [You94]. It should be noted that D,*

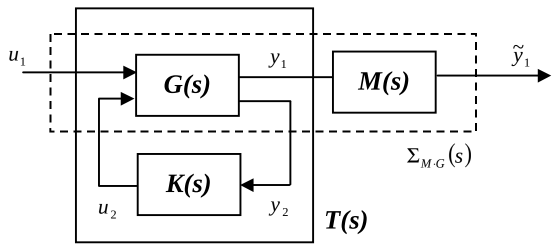

Figure 5.4: Robust Feedback with Multiplier

$G-K$ iteration still require "curve fitting" to generate that $D(s)$, $G(s)$ scales, and this deficiency may be overcome using the LMI based approach of [LSC94, BHPD94].

In this section, we briefly reviewed the $M-K$ iteration approaches to solve the μ/k_m-synthesis problem. We also described the benefits of $M-K$ iteration. However, there still has one intermediate multiplier factorization step that might introduce computational complexity and potential numerical problem. Therefore, the way to avoid this multiplier factorization step will be presented in next sections.

5.3 Problem Formulation

Consider the system shown in Figure 5.4. In this figure, $G(s)$ is a generalized plant:

$$G(s) := \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G21(s) & G_{22}(s) \end{bmatrix} \stackrel{ss}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \tag{5.6}$$

where $A \in \mathbf{R}^{n\times n}$, $D_{11} \in \mathbf{R}^{r\times r}$, $D_{22} \in \mathbf{R}^{s_2\times r_2}$, and $G_{ij} := D_{ij} + C_i(sI - A)^{-1}B_j$. Without loss of generality, we will set D_{22} to be zero matrix in this chapter. In the same figure, $K(s)$ is a linear time invaariant controller:

$$K(s) \stackrel{ss}{=} \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right], \text{ where } A_k \in \mathbf{R}^{k\times k},\ D_k \in \mathbf{R}^{r_2\times s_2},$$

and $M(s)$ is a generalized multiplier:

$$M(s) \stackrel{ss}{=} \left[\begin{array}{c|c} A_m & B_m \\ \hline C_m & D_m \end{array}\right], \text{ where } A_m \in \mathbf{R}^{m\times m},\ D_m \in \mathbf{R}^{r\times r}.$$

Then the state space form of the closed-loop system $T(s)$ and the weighted closed-loop system $M(s)T(s)$ can be expressed as following:

$$T(s) = \mathcal{F}_L\left\{G(s), K(s)\right\} \stackrel{ss}{=} \left[\begin{array}{cc|c} A & B_2C_k & B_1 \\ B_kC_2 & A_k & B_kD_{21} \\ \hline C_1 & D_{21}C_k & D_{11} \end{array}\right]$$

$$M(s)T(s) \stackrel{ss}{=} \left[\begin{array}{ccc|c} A_m & B_mC_1 & B_mD_{12}C_k & B_mD_{11} \\ 0 & A & B_2C_k & B_1 \\ 0 & B_kC_2 & A_k & B_kD_{21} \\ \hline C_m & D_mC_1 & D_mD_{12}C_k & D_mD_{11} \end{array}\right] = \left[\begin{array}{c|c} A_{MT} & B_{MT} \\ \hline C_{MT} & D_{MT} \end{array}\right].$$

We also consider the weighted plant of $\Sigma_{M\cdot G}(s)$ where

$$\Sigma_{M\cdot G}(s) \quad := \quad \left[\begin{array}{c|c} M(s) & 0 \\ \hline 0 & I \end{array}\right] G(s) \tag{5.7}$$

$$\stackrel{ss}{=}\left[\begin{array}{cc|c|c} A_m & B_mC_1 & B_mD_{11} & B_mD_{12} \\ 0 & A & B_1 & B_2 \\ \hline C_m & D_mC_1 & D_mD_{11} & D_mD_{12} \\ \hline 0 & C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c|c} A_\Sigma & B_{1\Sigma} & B_{2\Sigma} \\ \hline C_{1\Sigma} & D_{11\Sigma} & D_{12\Sigma} \\ \hline C_{2\Sigma} & D_{21\Sigma} & D_{22\Sigma} \end{array}\right]$$

and introduce the following two Riccati functions [SKS94]:

$$\begin{aligned} \mathrm{Ric1}(X) &:= A_u^T X + XA_u - XR_uX + Q_u \\ \mathrm{Ric2}(Y) &:= A_vY + YA_v^T - YR_vY + Q_v \end{aligned}$$

where

$$\begin{aligned}
A_u &= A_\Sigma - B_{1\Sigma}JC_{1\Sigma} - (B_{2\Sigma} - B_{1\Sigma}JD_{12\Sigma})J_{12}D_{12\Sigma}^T JC_{1\Sigma}, \\
Q_u &= \hat{C}_1^T\hat{C}_1, \\
R_u &= (B_{2\Sigma} - B_{1\Sigma}JD_{12\Sigma}))J_{12}(B_{2\Sigma} - B_{1\Sigma}JD_{12\Sigma}))^T - B_{1\Sigma}JB_{1\Sigma}^T, \\
A_v &= A_\Sigma - B_{1\Sigma}JC_{1\Sigma} - B_{1\Sigma}JD_{21\Sigma}^T J_{21}(C_{2\Sigma} - D_{21\Sigma}JC_{1\Sigma}), \\
Q_v &= \hat{B}_1\hat{B}_1^T, \\
R_v &= (C_{2\Sigma} - D_{21\Sigma}JC_{1\Sigma})^T J_{21}(C_{2\Sigma} - D_{21\Sigma}JC_{1\Sigma}) - C_{1\Sigma}^T JC_{1\Sigma}, \\
\hat{C}_1 &= (J - JD_{12\Sigma}J_{12}D_{12\Sigma}^T J)^{\frac{1}{2}}C_{1\Sigma}, \\
\hat{B}_1 &= B_{1\Sigma}(J - JD_{21\Sigma}^T J_{21}D_{21\Sigma}J)^{\frac{1}{2}}, \\
J &= (D_{11\Sigma} + D_{11\Sigma}^T)^{-1}, \\
J_{12} &= (D_{12\Sigma}^T JD_{12\Sigma})^{-1}, \\
J_{21} &= (D_{21\Sigma}JD_{21\Sigma}^T)^{-1}.
\end{aligned}$$

The following definition is referred to in our problem formulation.

Definition 5.1. *A square matrix transfer function $X(s)$ is generalized strongly positive real if:*

1. *$X(s)$ has no $j\omega$-axis poles.*

2. *$\exists \epsilon > 0 \ \ni \ \frac{1}{\epsilon} I > herm\{X(j\omega)\} > \epsilon I$ for all ω.*

The unstably-weighted robust control synthesis problem we will study is stated as flollows.

Problem Formulation Given an unstable matrix transfer function $M(s)$, find a controller satisfies the following requirements:

- The unstably-weighted closed-loop system $M(s)T(s)$ is generalized strongly positive real.
- The closed-loop system $T(s)$ is stable.

5.4 Unstably-Weighted Positive Real Riccati Inequalities

Positive real synthesis is concerned with finding a controller such that the closed-loop system is strongly positive real. Previous approaches to positive real synthesis can not be applied to multiplier synthesis problem directly when multiplier $M(s)$ is unstable, because $(A_\Sigma, C_{2\Sigma})$ is undetectable in the unstably-weighted control synthesis problem. Hence, multiplier factorization [CS92, SC93] plays an important role in multiplier synthesis. However, multiplier factorization restricts the selections of multiplier class and introduce computation complexity.

In this section, the author will introduce a two-Riccati Inequality solution for unstably-weighted robust control synthesis problem without applying multiplier factorization.

Theorem 5.1. *Consider the system $\Sigma_{M \cdot G}$ and suppose that $D_{12\Sigma}$ and $D_{21\Sigma}^T$ are full clumn rank, $D_{22\Sigma} = 0$ and $D_{11\Sigma} + D_{11\Sigma}^T > 0$. Then there exists a strictly proper* *controller $K(s)$ such that $M(s)T(s)$ is generalized strongly positive real and $T(s)$ is stable if and only if*

$$\exists X_I = X_I^T \text{ and } \begin{bmatrix} 0 & I_n \end{bmatrix} X_I^{-1} \begin{bmatrix} 0 \\ I_n \end{bmatrix} > 0 \ni Ric1(X_I) < 0, \tag{5.8}$$

$$\exists Y_I = Y_I^T \text{ and } \begin{bmatrix} 0 & I_n \end{bmatrix} Y_I \begin{bmatrix} 0 \\ I_n \end{bmatrix} > 0 \ni Ric2(Y_I) < 0, \tag{5.9}$$

$$X_I^{-1} - Y_I \geq 0. \tag{5.10}$$

Proof. From [And67, AM68], we know that herm$\{M(s)T(s)\} > 0$ if and only if there exists a noningular symmetric matrix Q such that

$$A_{MT}Q + QA_{MT}^T + (QC_{MT}^T - B_{MT})(D_{MT} + D_{MT}^T)^{-1}(QC_{MT}^T - B_{MT})^T < 0 \tag{5.11}$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{11} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix}$, Q_{11}, Q_{22} and Q_{33} are $m \times m$, $n \times n$ and $k \times k$ matrices respectively.

Obviously, when Q satisfied (5.11), it is routine to verify that

$$PA_{MT} + A_{MT}^T P + (C_{MT}^T - PB_{MT})(D_{MT} + D_{MT}^T)^{-1}(C_{MT}^T - PB_{MT})^T < 0$$

where $P = Q^{-1} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^T & P_{22} & P_{23} \\ P_{13}^T & P_{23}^T & P_{33} \end{bmatrix}$, P_{11}, P_{22} and P_{33} are $m \times m$, $n \times n$ and $k \times k$ matrices respectively.

Since the quadratic term in (5.11) is nonnegative definite, we have

$$\left[\begin{array}{c|cc} A_m & B_mC_1 & B_mD_{12}C_k \\ \hline 0 & A & B_2C_k \\ 0 & B_kC_2 & A_k \end{array}\right]\left[\begin{array}{c|cc} Q_{11} & Q_{12} & Q_{11} \\ \hline Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{array}\right] + \left[\begin{array}{c|cc} Q_{11} & Q_{12} & Q_{11} \\ \hline Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{array}\right]\left[\begin{array}{c|cc} A_m & B_mC_1 & B_mD_{12}C_k \\ \hline 0 & A & B_2C_k \\ 0 & B_kC_2 & A_k \end{array}\right]^T < 0 \tag{5.12}$$

Taking the (2,2) block of (5.12), it follows that

$$\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23}^T & Q_{33} \end{bmatrix}\begin{bmatrix} A & B_2C_k \\ B_kC_2 & A_k \end{bmatrix}^T + \begin{bmatrix} A & B_2C_k \\ B_kC_2 & A_k \end{bmatrix}\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23}^T & Q_{33} \end{bmatrix} < 0 \tag{5.13}$$

Form Lyapunov inequality, $T(s)$ is stable if and only if

$$\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23}^T & Q_{33} \end{bmatrix} > 0. \tag{5.14}$$

From [SKS94, THS95], there exists a strictly proper controller such that (3.4) is satisfied if and only if there exist nonsingular symmetric matrices X_I and Y_I such that $\text{Ric1}(X_I) < 0$ and $\text{Ric2}(Y_I) < 0$ where

$$X_I^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \quad \text{and} \quad Y_I^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}.$$

Since $\begin{bmatrix} Q_{22} & Q_{23} \\ Q_{23}^T & Q_{33} \end{bmatrix} > 0$ if and only if

$$Q_{22} > 0, \quad Q_{33} > 0 \text{ and } Q_{22} - Q_{23}Q_{33}^{-1}Q_{23}^T > 0 \tag{5.15}$$

It is obvious that

$$\begin{bmatrix} 0 & I_n \end{bmatrix} X_I^{-1} \begin{bmatrix} 0 \\ I_n \end{bmatrix} = Q_{22} > 0 \tag{5.16}$$

and

$$X^{-1} - Y_I = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Q_{13} \\ Q_{23} \end{bmatrix} Q_{33}^{-1} \begin{bmatrix} Q_{13}^T & Q_{23}^T \end{bmatrix} \geq 0 \tag{5.17}$$

Thus, we obtain (5.8) and (5.10).

Taking the (2.2) block of (5.17), we get

$$\begin{bmatrix} 0 & I_n \end{bmatrix} Y_I \begin{bmatrix} 0 \\ I_n \end{bmatrix} = Q_{22} - Q_{23}Q_{33}^{-1}Q_{23}^T > 0 \tag{5.18}$$

From (5.15), we obtain (5.9). □

Remark 5.2. *The proof presented in [Hua96] was positive real LMI formulae to develop analogious LMI formulae for the unstably-weighted control synthesis problem. WIth the full conlumn rank assumption of $D_{12\Sigma}$ and $D_{21\Sigma}^T$, the two Riccati inequalities of unstably-weighted control system can be derived from its corresponding LMI formulae.*

5.5 Unstably-Weighted Positive Real Riccati Equations

In the previous section we have given the necessary and sufficient conditions for the existence of solutions to the unstably-weighted control synthesis problem in terms of algebraic Riccati inequalities (ARI). In this section, we reduce the result in Theorem 5.1 to the solutions of algebraic Riccati equations (ARE). Two extra assumptions are needed:

1. The matrix $\begin{bmatrix} A_\Sigma - j\omega I & B_{2\Sigma} \\ C_{1\Sigma} & D_{12\Sigma} \end{bmatrix}$ has full column rank for all $\omega \in [0, \infty)$
2. The matrix $\begin{bmatrix} A_\Sigma - j\omega I & B_{1\Sigma} \\ C_{2\Sigma} & D_{21\Sigma} \end{bmatrix}$ has full row rank for all $\omega \in [0, \infty)$

These preliminary conditions are analogous to the corresponding conditions in the $LQG/\mathcal{H}_\infty$ cases.

Before we reduce the necessary and sufficient conditions in Theorem 5.1 to the solutions of ARE. We have to establish certain relations between the solutions of ARI and the solutions of the corresponding ARE. Therefore the following lemmas are referred to in Theorem 5.2.

Lemma 5.1. *Let X_I be an invertible solution for Ric1$(X_I) < 0$ and $(-A_u, \hat{C}_1)$ is detectable, then there exists a unique solution Q_E for $Q_E A_u^T + A_u Q_E - R_u + Q_E \hat{C}_1^T \hat{C}_1 Q_E = 0$ such that $-A_u^T - \hat{C}_1^T \hat{C}_1 Q_E$ is strictly Hurwitz and $Q_E - X_I^{-1} > 0$. Moreover, if Q_E is nonsingular, then there exists a unique solution $X_E = Q_E^{-1}$ for Ric1$(X_E) = 0$ such that $A_u - R_u X_E$ is strictly Hurwitz.*

Proof. It is obvious that we can use the congruence transformation to get

$$X_I^{-1} Ric1(X_I) X_I^{-1} = X_I^{-1} A_u^T + A_u X_I^{-1} - R_u + X_I^{-1} \hat{C}_1^T \hat{C}_1 X_I^{-1} < 0.$$

From [Fai87, RV88], the ARE corresponds to the above ARI is

$$Q_E(-A_u^T) + (-A_u)Q_E - Q_E(\hat{C}_1^T\hat{C}_1)Q_E + R_u = 0$$

where Q_E is the solution for the above ARE. Since $(-A_u, \hat{C}_1)$ is detectable, we can use the result in [Fai87, RV88] to conclude that the equation

$$Q_E A_u^T + A_u Q_E - R_u + Q_E(\hat{C}_1^T\hat{C}_1)Q_E = 0 \tag{5.19}$$

has a solution Q_E such that $Q_E - X_I^{-1} > 0$ and $-A_u^T - \hat{C}_1^T\hat{C}_1 Q_E$ is strictly Hurwitz. Moreover, if Q_E is nonsingular, we can premultiply (5.19) by $X_E = Q_E^{-1}$ and postmultiply by X_E to obtain

$$\text{Ric1}(X_E) = X_E A_u + A_u^T X_E - X_E R_u X_E + \hat{C}_1^T\hat{C}_1 = 0.$$

Since X_E is invertible, we can postmultiply X_E^{-1} to $\text{Ric1}(X_E) = 0$ and obtain

$$X_E(A_u - R_u X_E)X_E^{-1} = -A_u^T - \hat{C}_1^T\hat{C}_1 Q_E$$

Hence, $A_u - R_u X_E$ is strictly Hurwitz.

□

Lemma 5.2. *Let Y_I be an invertible solution for Ric2$(Y_I) < 0$, $\begin{bmatrix} 0 & I_n \end{bmatrix} Y_I \begin{bmatrix} 0 \\ I_n \end{bmatrix} > 0$ and there exists an unique solution Y_E for Ric2$(Y_E) = 0$ such that $A_v^T - R_v Y_E$ is strictly Hurwitz, then $Y_I - Y_E > 0$.*

Proof. Without loss of generality, we assume $D_{11\Sigma} + D_{11\Sigma}^T = I$, then

$$\mathrm{Ric2}(Y_I) = A_v Y_I + Y_I A_v^T - Y_I R_v Y_I + \hat{B}_1 \hat{B}_1^T < 0$$

where

$$\begin{aligned} A_v &= \begin{bmatrix} A_m & B_m C_1 \\ 0 & A \end{bmatrix} - \begin{bmatrix} B_m D_{11} \\ B_1 \end{bmatrix} \begin{bmatrix} C_m & D_m C_1 \end{bmatrix} \\ &\quad - \begin{bmatrix} B_m D_{11} \\ B_1 \end{bmatrix} D_{21}^T J_{21} (\begin{bmatrix} 0 & C_2 \end{bmatrix} - D_{21} \begin{bmatrix} C_m & D_m C_1 \end{bmatrix}) \\ \hat{B}_1 &= \begin{bmatrix} B_m D_{11} \\ B_1 \end{bmatrix} (I - D_{21}^T J_{21} D_{21}). \end{aligned} \tag{5.20}$$

Applying a similarity transformation on the state space form of $G(s)$, we obtain the following decompositions

$$\hat{B}_1 = \begin{bmatrix} * \\ B_1(I - D_{21}^T J_{21} D_{21}) \end{bmatrix} = \left[\begin{array}{c} * \\ \hline B_{11} \\ 0 \end{array}\right] = \left[\begin{array}{c} * \\ B_{11} \\ \hline 0 \end{array}\right] = \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix}$$

$$\begin{aligned} A_v &= \begin{bmatrix} * & * \\ -B_1(I - D_{21}^T J_{21} D_{21}) C_m & A - B_1 D_m C_1 - B_1 D_{21}^T J_{21} (C_2 - D_{21} D_m C_1) \end{bmatrix} \\ &= \left[\begin{array}{c|cc} * & * & * \\ \hline -B_{11} C_m & A_{11} & A_{12} \\ 0 & 0 & A_{22} \end{array}\right] = \left[\begin{array}{cc|c} * & * & * \\ -B_{11} C_m & A_{11} & A_{12} \\ \hline 0 & 0 & A_{22} \end{array}\right] = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & A_{22} \end{bmatrix} \end{aligned}$$

where A_{22} is stable and $(-\hat{A}_{11}, \hat{B}_{11})$ is stabilizable.

Let

$$S := \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} := Y_I^{-1} := \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} (Y_{11} - Y_{12}Y_{22}^{-1}Y_{12}^T)^{-1} & * \\ & * \end{bmatrix}$$

and then premultiply Y_I^{-1} and postmultiply Y_I^{-1} on $\text{Ric2}(Y_I)$, we get

$$\begin{aligned} Y_I^{-1}\text{Ric2}(Y_I)Y_I^{-1} &= \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & A_{22} \end{bmatrix} + \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & A_{22} \end{bmatrix}^T \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \\ &+ \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix} \begin{bmatrix} \hat{B}_{11}^T & 0 \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \\ &- \begin{bmatrix} R_{v11} & R_{v12} \\ R_{v12}^T & R_{v22} \end{bmatrix} < 0 \end{aligned} \tag{5.21}$$

where $R_v = \begin{bmatrix} R_{v11} & R_{v12} \\ R_{v12}^T & R_{v22} \end{bmatrix}$.

Taking the (1,1) block of (5.21), it follows that

$$S_{11}\hat{A}_{11} + \hat{A}_{11}^T S_{11} + S_{11}\hat{B}_{11}\hat{B}_{11}^T S_{11} - R_{v11} < 0$$

From Lemma 5.1, there exists a unique solution S_{11E} such that

$$S_{11E}\hat{A}_{11} + \hat{A}_{11}^T S_{11E} + S_{11E}\hat{B}_{11}\hat{B}_{11}^T S_{11E} - R_{v11} = 0,$$

$-\hat{A}_{11}^T - \hat{B}_{11}\hat{B}_{11}^T S_{11E}$ is strictly Hurwitz and $S_{11E} > S_{11}$. Since there exists a unique solution Y_E for $\text{Ric2}(Y_E)$ such that $A_v^T - R_v Y_E$ is strictly Hurwitz.

We can conclude that S_{11E} is invertible and $Y_E = \begin{bmatrix} S_{11E}^{-1} & 0 \\ 0 & 0 \end{bmatrix}$ is the unique solution for $\text{Ric2}(Y_E) = 0$ such that $A_v^T - R_v Y_E$ is strictly Hurwitz.

Now we verify that Y_E will make $A_v^T - R_v Y_E$ strictly Hurwitz.

$$A_v^T - R_v Y_E = \begin{bmatrix} A_{11}^T - R_{v11} S_{11E}^{-1} & 0 \\ & A_{22}^T \end{bmatrix} = \begin{bmatrix} S_{11E}(-\hat{A}_{11} - \hat{B}_{11}\hat{B}_{11}^T S_{11E}) S_{11E}^{-1} & 0 \\ & A_{22}^T \end{bmatrix}$$

where $-\hat{A}_{11}^T - \hat{B}_{11}\hat{B}_{11}^T S_{11E}$ and A_{22} are strictly Hurwitz. Hence Y_E is a stabilizing solution for $\text{Ric2}(Y_E) = 0$.

Since $S_{11}^{-1} - S_{11E}^{-1} > 0$ and $\begin{bmatrix} 0 & I_n \end{bmatrix} Y_I \begin{bmatrix} 0 \\ I_n \end{bmatrix} = Y_{22} > 0$, it can be shown that

$$\begin{aligned} Y_I - Y_E &= \begin{bmatrix} Y_{11} - S_{11E}^{-1} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} = \begin{bmatrix} (S_{11}^{-1} + Y_{12}Y_{22}^{-1}Y_{12}^T) - S_{11E}^{-1} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix} \\ &= \begin{bmatrix} I & Y_{12}Y_{22}^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} S_{11}^{-1} - S_{11E}^{-1} & 0 \\ 0 & Y_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ Y_{22}^{-1}Y_{12}^T & I \end{bmatrix} > 0 \end{aligned}$$

where $Y_I = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^T & Y_{22} \end{bmatrix}$. □

After establishing the relations between X_I, X_E and Y_I, Y_E, we can now reduce the results of the Theorem 5.1 to solutions of the corresponding AREs. In the following theorem, we assume $(-A_u, \hat{C}_1)$ is detectable.

Theorem 5.2. *Consider the system $\Sigma_{M \cdot G}$ and suppose that stabilizing solution X_E and Y_E for Ric1$(X_E) = 0$ and Ric2$(Y_E) = 0$ exist. Then there exists a strictly proper* *controller $K(s)$ such that $M(s)T(s)$ is generalized strongly positive real and $T(s)$ is stable if and only if X_E and Y_E satisfy*

$$\begin{bmatrix} 0 & I_n \end{bmatrix} X_E^{-1} \begin{bmatrix} 0 \\ I_n \end{bmatrix} \geq 0, \tag{5.22}$$

$$\begin{bmatrix} 0 & I_n \end{bmatrix} Y_E \begin{bmatrix} 0 \\ I_n \end{bmatrix} \geq 0, \tag{5.23}$$

$$X_E^{-1} - Y_E > 0 \tag{5.24}$$

Moreover, when these conditions are satisfied, there exists a strictly proper controller $K(s) = C_k(sI - A_k)^{-1}B_k$ where

$$\begin{aligned} B_k &= (I - Y_E X_E)^{-1}(Y_E C_{2\Sigma}^T + (B_{1\Sigma} - Y_E C_{1\Sigma}^T) J D_{21\Sigma}^T) J_{21}, \\ C_k &= -J_{12}(B_{2\Sigma}^T X_E + D_{12\Sigma}^T J (C_{1\Sigma} - B_{1\Sigma}^T X_E)) \\ A_k &= A_\Sigma + B_{2\Sigma} C_k - B_k C_{2\Sigma} - (B_{1\Sigma} - B_k D_{21\Sigma})(C_{1\Sigma} - B_{1\Sigma}^T X_E + D_{12\Sigma} C_k) \end{aligned} \tag{5.25}$$

reders $M(s)T(s)$ generalized strongly positive real and $T(s)$ stable.

Proof. herm $\{M(s)T(s)\} > 0$ if and only if there exists a symmetric matrix Q such that

$$A_{MT}Q + QA_{MT}^T + (QC_{MT}^T - B_{MT})(D_{MT} + D_{MT}^T)^{-1}(QC_{MT}^T - B_{MT})^T = 0 \tag{5.26}$$

Since the quadratic term in (5.26) is nonnegative definite, we have

$$\left[\begin{array}{c|cc} A_m & B_mC_1 & B_mD_{12}C_k \\ \hline 0 & A & B_2C_k \\ 0 & B_kC_2 & A_k \end{array}\right] Q + Q \left[\begin{array}{c|cc} A_m & B_mC_1 & B_mD_{12}C_k \\ \hline 0 & A & B_2C_k \\ 0 & B_kC_2 & A_k \end{array}\right]^T \leq 0. \quad (5.27)$$

Taking the (2,2) block of (5.27), it follows that

$$\begin{bmatrix} 0 & I_{n+k} \end{bmatrix} Q \begin{bmatrix} 0 \\ I_{n+k} \end{bmatrix} \begin{bmatrix} A & B_2C_k \\ B_kC_2 & A_k \end{bmatrix}^T + \begin{bmatrix} A & B_2C_k \\ B_kC_2 & A_k \end{bmatrix} \begin{bmatrix} 0 & I_{n+k} \end{bmatrix} Q \begin{bmatrix} 0 \\ I_{n+k} \end{bmatrix} \leq 0. \quad (5.28)$$

Form Lyapunov inequality, $T(s)$ is stable if and only if

$$\begin{bmatrix} 0 & I_{n+k} \end{bmatrix} Q \begin{bmatrix} 0 \\ I_{n+k} \end{bmatrix} \geq 0. \quad (5.29)$$

From Lemma 5.1 and from the existence assumption of the stabilizing solution X_E for $\mathrm{Ric1}(X_E) = 0$, we can conclude that X_E is nonsingular and $X_E^{-1} - X_I^{-1} > 0$. Form [SKS94], we can derive that $\begin{bmatrix} I_{n+m} & 0 \end{bmatrix} Q \begin{bmatrix} I_{n+m} \\ 0 \end{bmatrix} = X_E^{-1}$. Therefore, we obtain condition (5.22).

From Lemma 5.2 and from the existence assumption of the stabilizing solution Y_E for $\mathrm{Ric2}(Y_E) = 0$, we can conclude that $Y_I - Y_E > 0$. Also, $\mathrm{Ric2}(Y_E)$ can be expressed as following:

$$\begin{aligned} \mathrm{Ric2}(Y_E) &= (A_\Sigma + LC_{2\Sigma})Y_E + Y_E(A_\Sigma + LC_{2\Sigma})^T \\ &\quad + (B_{1\Sigma} + L\bar{D}_{21\Sigma} - Y_E\bar{C}_{1\Sigma}^T)(B_{1\Sigma} + L\bar{D}_{21\Sigma} - Y_E\bar{C}_{1\Sigma}^T)^T = 0 \end{aligned} \quad (5.30)$$

where $L = -(Y_E C_{2\Sigma}^T + (B_{1\Sigma} - Y_E B_{1\Sigma}^T) J D_{21\Sigma}^T) J_{21}$.

Since the quadratic term in (5.30) is nonnegative definite, we have

$$(A_\Sigma + LC_{2\Sigma})Y_E + Y_E(A_\Sigma + LC_{2\Sigma})^T = (\begin{bmatrix} A_m & C_1B_m \\ 0 & A \end{bmatrix} + L\begin{bmatrix} 0 & C_2 \end{bmatrix})Y_E \quad (5.31)$$
$$+Y_E(\begin{bmatrix} A_m & C_1B_m \\ 0 & A \end{bmatrix} + L\begin{bmatrix} 0 & C_2 \end{bmatrix})^T$$
$$\leq 0.$$

Taking the (2,2) block of (5.31), we obtain

$$(A + L_2C_2)Y_{22E} + Y_{22E}(A + L_2C_2)^T \leq 0$$

where $Y_E = \begin{bmatrix} Y_{11E} & Y_{12E} \\ Y_{12E}^T & Y_{22E} \end{bmatrix}$ and $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$. From Lyapunov inequality, $A + L_2C_2$ is stable if and only if $Y_{22E} \geq 0$. Therefore we obtain (5.23).

Since $X_E^{-1} - X_I^{-1} > 0$, $Y_I - Y_E > 0$ and $X_I^{-1} - Y_I \geq 0$ from (5.10), we obtain

$$X_E^{-1} - Y_E = (X_E^{-1}X_I^{-1}) + (X_I^{-1} - Y_I) + (Y_I - Y_E) > 0.$$

Therefore we obtain (5.24). Moreover, A_k, B_k and C_k can be obtained by the similar techniques in [SKS94]. □

Remark 5.3. *In positive real control problem, the necessary and sufficient conditions for the existence of the controller are that the positive real Riccati equation solutions X_E, Y_E are positive semidefinite and $\bar{\sigma}(X_EY_E) < 1$. The result in Theorem 5.2 is an extension to conventional positive real control problem except we check the positive semidefinite properties of $\begin{bmatrix} 0 & I_n \end{bmatrix} X_E^{-1} \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ and $\begin{bmatrix} 0 & I_n \end{bmatrix} Y_E \begin{bmatrix} 0 \\ I_n \end{bmatrix}$ instead.*

Notice that the results in Theorem 5.2 are stated with the assumption $(-A_u, \hat{C}_1)$ detectable for the existence of X_E^{-1}. If this assumption does not hold, the stabilizing solution of the first ARE will be singular. In Chapter 7 of [Hua96] a perturbation method is introduced for the singular X_E case.

Another approach is to use the "third" Riccati equation solution for $\mathcal{H}_\infty$ synthesis problem which has been introduced in [DGKF89, PAJ91]. Applying the techniques similar to those employed in [PAJ91], a "third" Riccati equation for positive real synthesis is developed. Therefore, we may refine the two ARE solution for unstably-weighted control problem presented in Theorem 5.2 by applying the third Riccati equation solution Z_E. The result in the following theorem requires no perturbation method when X_E is singular. We first define

$$\mathrm{Ric3}(Z) = Z(A_u - Y_E Q_u) + (A_u - Y_E Q_u)^T Z - Z(R_{2Z} - R_{1Z})Z + Q_u$$

where

$$\begin{aligned} R_{1Z} &= (Y_E(C_{2\Sigma} - D_{21\Sigma} J C_{1\Sigma})^T + B_{1\Sigma} J D_{21\Sigma}^T) J_{21} (Y_E(C_{2\Sigma} - D_{21\Sigma} J C_{1\Sigma})^T + B_{1\Sigma} J D_{21\Sigma}^T)^T, \\ R_{2Z} &= ((Y_E C_{1\Sigma}^T - B_{1\Sigma}) J D_{12\Sigma} + B_{2\Sigma}) J_{12} ((Y_E C_{1\Sigma}^T - B_{1\Sigma}) J D_{12\Sigma} + B_{2\Sigma})^T \end{aligned}$$

and Y_E is the stabilizing solution for $\mathrm{Ric2}(Y_E) = 0$.

Theorem 5.3. *Consider the system $\Sigma_{M \cdot G}$ and suppose that stabilizing solutions Y_E and Z_E for Ric2(Y_E) $= 0$ and Ric3(Z_E) $= 0$ exist. Then there exists a strictly proper controller $K(s)$ such that herm$\{M(s)T(s)\} > 0$ and $T(s)$ is stable if and only if*

$$\begin{bmatrix} 0 & I_n \end{bmatrix} Y_E \begin{bmatrix} 0 \\ I_n \end{bmatrix} \geq 0, \qquad (5.32)$$

$$Z_E \geq 0. \tag{5.33}$$

Moreover, when these conditions are satisfied, there exists a strictly proper controller $K(s) = C_k(sI - A_k)^{-1}B_k$ *where*

$$\begin{aligned}
B_k &= (I + Y_E Z_E)(Y_E C_{2\Sigma}^T + (B_{1\Sigma} - Y_E C_{1\Sigma}^T) J D_{21\Sigma}^T) J_{21}, \\
C_k &= -J_{12}((B_{2\Sigma}^T X_E - D_{12\Sigma}^T J B_{1\Sigma}^T) Z_E (I + Y_E Z_E)^{-1} + D_{12\Sigma}^T J C_{1\Sigma}), \\
A_k &= A_\Sigma + B_{2\Sigma} C_k - B_k C_{2\Sigma} - (B_{1\Sigma} - B_k D_{21\Sigma})(C_{1\Sigma} - B_{1\Sigma}^T Z_E (I + Y_E Z_E)^{-1} + D_{12\Sigma} C_k)
\end{aligned} \tag{5.34}$$

renders $M(s)T(s)$ *generalized strongly positive real and* $T(s)$ *stable.*

Proof. Since $Y_I - Y_E > 0$ and $X_I - X_I Y_I X_I \geq 0$, it is obvious that $X_I - X_I Y_E X_I > 0$. Premultiply $(I - X_I Y_E)^{-1}$ and postmultiply its transpose $(I - X_I Y_E)^{-1}$, we obtain $(I - X_I Y_E)^{-1} X_I > 0$. We now define

$$Z_I := (I - X_I Y_E)^{-1} X_I > 0.$$

Moreover, it follows from the definition of Z_I that

$$X_I = Z_I (I + Y_E Z_I)^{-1} = (I + Z_I Y_E)^{-1} Z_I. \tag{5.35}$$

In order to obtain condition (5.33), we define

$$\begin{aligned}
\mathrm{Ric3}(Z) &:= (I + Z Y_E)\mathrm{Ric1}(X)(I + Y_E Z) + Z \mathrm{Ric2}(Y_E) Z \\
&= Z(A_u - Y_E Q_u) + (A_u - Y_E Q_u)^T Z - Z(R_{2Z} - R_{1Z}) Z + Q_u
\end{aligned}$$

where $X = Z(I + Y_E Z)^{-1}$. Substituting (5.35) into $\mathrm{Ric1}(X_I) < 0$ and using $\mathrm{Ric2}(Y_E) = 0$, it is straightforward to verify that $Z_I > 0$ satisfies $\mathrm{Ric3}(Z_I) < 0$ where $X = X_I$. We can also show that

$$\begin{bmatrix} 0 & I_n \end{bmatrix} X_I^{-1} \begin{bmatrix} 0 \\ I_n \end{bmatrix} = \begin{bmatrix} 0 & I_n \end{bmatrix} (Z_I^{-1} + Y_E) \begin{bmatrix} 0 \\ I_n \end{bmatrix} > 0$$

since $Z_I > 0$ and $\begin{bmatrix} 0 & I_n \end{bmatrix} Y_E \begin{bmatrix} 0 \\ I_n \end{bmatrix} \geq 0$.

Applying the similar technique in the proof of Theorem 5.1 in [SKS94], we can show that there exists a solution $Z_E \geq 0$ such that $\mathrm{Ric3}(Z_E) = 0$ and $Z_I - Z_E > 0$ are satisfied. Hence we obtain (5.33). Moreover, A_k, B_k and C_k can be obtained by the similar techniques in [DGKF89, SKS94] where

$$\begin{aligned} B_k &= (I - Y_E X_E)^{-1}(Y_E C_{2\Sigma}^T + (B_{1\Sigma} - Y_E C_{1\Sigma}^T) J D_{21\Sigma}^T) J_{21}, \\ C_k &= -J_{12}(B_{2\Sigma}^T X_E + D_{12\Sigma}^T J (C_{1\Sigma} - B_{1\Sigma}^T X_E)) \\ A_k &= A_\Sigma + B_{2\Sigma} C_k - B_k C_{2\Sigma} - (B_{1\Sigma} - B_k D_{21\Sigma})(C_{1\Sigma} - B_{1\Sigma}^T X_E + D_{12\Sigma} C_k), \\ X_E &= Z_E (I + Y_E Z_E)^{-1} \end{aligned} \tag{5.36}$$

and we obtain (5.34).

Remark 5.4. *The results of Theorem 5.3 show that by introducing the "third" Riccati equation, the assumption of* $(-A_u, \hat{C}_1)$ *detectable can be discarded. This approach also reduces the necessary and sufficient conditions from three to two which simplifies the previous results.*

5.6 Conclusion

This chapter studies the necessary and sufficient conditions for solving the unstably-weighted robust control synthesis problems. It turns out that those conditions can be interpreted in terms of the solutions of two algebraic Riccati inequalities or two algebraic Riccati equations. The results in this chapter open the way to directly design a robust controller without having to first compute equivalent stable weighting functions via the multiplier factorization approaches.

However, the existence assumption of the stabilizing solution for unstably-weighted positive real Riccati equation i.e., $\mathrm{Ric2}(Y_E) = 0$ is stringent. It seems that a proper selection of the corresponding Hamiltonian matrix eigenspace, i.e., the selection of unstabilizing Riccati equation solution, might offer an alternative way to solve the unstably-weighted robust control synthesis problems.

Chapter 6

Conclusion

6.1 Summary

The research recorded in this dissertation is mainly focused on the stability analysis and robust control synthesis with the generalized multipliers. For stability analysis with generalized multipliers, we start by linking the conic sectors of the topological separation framework to the integral quadratic constraints of the IQC framework. In Chapter 2 we analyze the role of quadratic separating functionals used in both the topological separation framework and integral quadratic constraints (IQC) framework for stability analysis. The forms of quadratic separating functionals that are useful for establishing topological separation are presented (cf. Theorem 2.2).

Next, working from the quadratic separating functionals used for stability and robustness analysis of the feedback systems, the results of Chapter 3 show that a stable, minimum phase state space factors always exists for the quadratic constraints in the topological framework and IQCs framework for robust analysis. As a consequence, the existence of stable and stability invertible sector transformation form of spectral factors is proved (cf. Lemma 3.2, Theorem 3.1).

In Chapter 4 we considered the generalized multipliers used in the IQC framework. We first showed the existence of canonical factorization of the generalized positive real (GPR) multipliers (cf. Theorem 4.3). Based on the canonical factorization results, we showed that the quadratic constraints with the generalized positive real multipliers in the IQCs framework can be unimodularly congruent to an indefinite constant matrix and the stable, minimum phase factor has a diagonal form (cf. Theorem 4.4).

For the robust control synthesis with generalized multipliers, we formulated a new unstably-weighted robust control synthesis problem in Chapter 5. In this problem formulation, we can obtain a robust controller without having to factorize the generalized multipliers, i.e., the unstably-weighting matrix. Based on the positive real control approach, we presented Riccati inequalities and Riccati equation solution for the unstably-weighted control synthesis problem (cf. Theorem 5.1, Theorem 5.2, Theorem 5.3).

6.2 Future Directions

Along this line of research, we have the following further research topics:

- Develop the matrix inequality conditions for canonical factorization of rational transfer function. This is because for the rational transfer function to have a canonical factorization, it must satisfy a complementary condition. Although this condition is very explicit, it is somewhat hard to incorporate in numerical algorithms, and hence we need to identify conditions that are easier to check numerically.

- Apply the generalized multiplier factorization results to the unstably-weighted robust control problem to see if we can get further inside of the unstably-weighted robust control synthesis problem.

- Develop an algorithm to select the proper ARE solutions to satisfy the necessary and sufficient ARE conditions for the unstably-weighted robust control synthesis problem.

- Develop modified formulae for constructing the stabilizing controller for the unstably-weighted robust control synthesis problem.

- Develop an LMI synthesis approach to the unstably-weighted robust control synthesis problem.

- Analyze the relation between the matrix pencil theory and the unstably-weighted robust control problem. Establish the connections between the non-causal multiplier and the stability of the closed-loop system.

- Demonstrate the application of the unstably-weighted robust control synthesis problem. Compare the results with standard $D - K$ or $M - K$ iteration in μ/k_m-synthesis.

This dissertation is concluded by summarizing the research presented, and by proposing an outline for future work on the problems of stability analysis and robust control synthesis with generalized multipliers.

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