*Proc. American Control Conf.*, Chicago, IL, June 28–30, 2000.

# Zames-Falb Multipliers for MIMO Nonlinearities[1]

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**Abstract**

In their celebrated 1968 paper on nonlinear stability, Zames and Falb determined a class of multipliers that preserve positivity of monotone SISO nonlinearities. They conjectured that their results might also hold for incrementally positive, norm-bounded MIMO nonlinearities. In this note, we demonstrate that their conjecture regarding MIMO nonlinearities holds true only if a further restriction is applied. Specifically, we show that it suffices either to restrict the nonlinearity to be the gradient of a convex real-valued function or to restrict the multiplier to be a real-valued function of frequency.

## 1 Introduction

A class of non-causal multipliers was introduced by Zames and Falb in [1] to study the stability of the system shown in Fig. 1 having a stable, linear time-invariant (LTI) plant $H$ in the feed-forward path and a memoryless, monotone nonlinearity $N$ in the feedback path (see Figure 1). Briefly speaking, the Zames-Falb multiplier approach to determining stability of a system rests on finding a class $\mathcal{M}$ of possibly non-causal, linear-time-invariant multipliers that is *positivity preserving* for $N$ in the sense that $M \in \mathcal{M}$ implies positivity of the operator $M^*N$. Additionally, the multipliers $M \in \mathcal{M}$ are required to be factorizable as

$$M = M_- M_+; \tag{1}$$

where $M_-, M_+$ have the properties that

$$M_-, M_+ \text{ are invertible;} \tag{2}$$

$$M_+, M_+^{-1}, M_-^*, M_-^{*-1} \text{ causal with finite gain.} \tag{3}$$

These properties ensure that for any such multiplier, stability of the system shown in Figure 1 is equivalent to that of the system shown in Figure 2. Stability of the system then follows if (see Theorem 2 of [1]) $MH$ is strongly positive and $N$ has finite gain. In [1], results were proved for single-input single-output (SISO) nonlinearity $N$ and it was claimed, as a concluding remark, that in the multivariable case

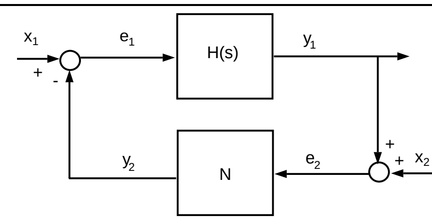


**Figure 1:** The feedback interconnection has a stable, LTI plant $H(s)$ and a memoryless nonlinearity $N$.

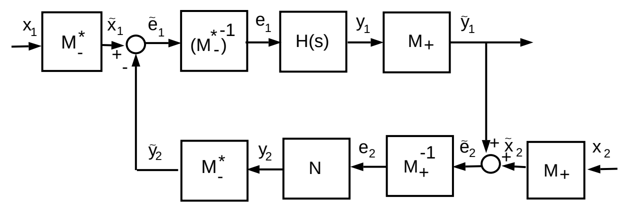


**Figure 2:** Equivalent transformed feedback interconnection with multipliers. The multipliers $M_+, M_+^{-1}, M_-^*, M_-^{*-1}$ are causal and stable with finite gain.

> "*. . . the mapping $N$ is no longer defined by a scalar function $N$ but by a vector function $N_v$ with the properties:*
>
> $$N_v(0) = 0; \tag{4}$$
>
> $$\langle r - s, N_v(r) - N_v(s)\rangle \ \geq 0; \tag{5}$$
>
> *there is some constant $c \geq 0$ such that*
>
> $$||N_v(r)|| \leq c||r|| \text{ for all } r. \tag{6}$$
>
> *A careful perusal of our proofs will clearly indicate the validity of this generalization.*"
> **Zames and Falb [1]**

In fact, this claim is incorrect without further restrictions on $N_v$, as we shall show in Section 2.

Recently, extensions of the Zames-Falb results to the special case of systems which have multiple *scalar* nonlinearities, every one of which is described by (4)–(6) modulo the loop shifting transformation, have been investigated (see, e.g., [2], [3], [4] and references therein). However, literature on multipliers for the general case (i.e. non-diagonal MIMO nonlinearities)

[1] This research is supported by AFOSR F49620-98-1-0026.
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[3] Vishwesh Kulkarni thanks Myungsoo Jun (USC) and Prof. Stephan Bohacek (USC) for the relevant discussions.

is sparse. While re-examining positivity of the scalar operators relevant to the system shown in Fig. 1 in SISO setting, Willems [5] acknowledged that difficulties might exist in extending some of the results, e.g. Lemma 8 of [1], to MIMO case (see [5, Chapter 3, pp. 66]) and the question whether the Zames-Falb claim for the MIMO case is true or false has remained unsettled thus far. In this note, we show that such a multiplier *can* fail to preserve positivity of the MIMO nonlinearity. However, if the MIMO nonlinearity is additionally stipulated to be the gradient of a potential function, it can be shown that the Zames-Falb multipliers do indeed preserve its positivity.

The paper is organized as follows. We begin with a simple counterexample to the Zames-Falb conjecture in Section 2. Notation and a preliminary lemma are introduced in Section 3. The problem formulation is given in Section 4. Main results are derived in Section 5 and briefly discussed in Section 6. Conclusions are in Section 7. Relevant background results may be found in the Appendix.

## 2 MIMO Counterexample

Consider the trivial *linear* nonlinearity $N_v : \mathbb{R}^2 \to \mathbb{R}^2$

$$N_v(x_1, x_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (7)$$

Nonlinearity (7) satisfies (4)–(6), yet the operator $M^*N_v$ fails to be positive for any Zames-Falb multiplier having a non-zero imaginary part, $\mathrm{Im}(\widehat{m})(j\omega)) \neq 0$. Indeed, the Hermitian part of its frequency-response matrix has a negative eigenvalue

$$\lambda_{min}\left(\mathrm{herm}\left(\widehat{m}^*(j\omega)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\right) = -|\mathrm{Im}(\widehat{m}(j\omega))| < 0$$

whence, by Lemma A1, $M^*N_v$ cannot be a positive operator. ■

A careful scrutiny of the proofs in [1] reveals the source of the difficulty in extending the Zames-Falb proofs to the MIMO case. The problem arises due to the fact that a preliminary result used to establish positivity of $M^*N$, Lemma 7 of [1], states

$$xN(x) - yN(x) \geq P(x) - P(y) \qquad \forall x, y \in \mathbb{R} \quad (8)$$

where $P(x) \doteq \int_0^x N(\zeta)d\zeta$. In the MIMO case $x, y \in \mathbb{R}^n$, the integral in (8) is path-dependent if (and only if) $\mathrm{skew}(\frac{\partial N_v(x)}{\partial x}) \neq 0$ (see Lemma A4 in Appendix). Therefore, the potential function $P(x)$ *may not even exist* so that the Lemma 7 [1] argument is invalid for the MIMO case.

## 3 Preliminaries

Notation used is summarized in Table 1.

**Table 1:** Notation

| Symbol | Meaning |
|---|---|
| $\mathbb{R}$ | Set of all real numbers |
| $\mathbb{C}$ | Set of all complex numbers |
| $x, y$ | Real signals — possibly vector-valued or matrix valued. |
| $F, G$ | Operators. |
| $\overline{(\cdot)}^T$ | Conjugate transpose of vector or matrix $(\cdot)$. |
| $\mathrm{herm}(m)$ | $= \frac{1}{2}(m + \overline{m}^T)$, for $m \in \mathbb{C}^{n\times n}$ or $\mathbb{R}^{n\times n}$. |
| $\mathrm{skew}(m)$ | $= \frac{1}{2}(m - \overline{m}^T)$, for $m \in C^{n\times n}$ or $\mathbb{R}^{n\times n}$. |
| $\mathrm{trace}[m]$ | $= \sum_i m_{ii}$, (trace of square matrix $m$). |
| $\lambda_{min}(m)$ | least eigenvalue of matrix $\mathrm{herm}(m)$ |
| $P_\tau$ | $[P_\tau x](t) = \begin{cases} x(t), & \text{if } -\tau \leq t \leq \tau \\ 0, & \text{otherwise} \end{cases}$ (2-sided time-truncation). |
| $\langle x, y\rangle$ | $= \int_{-\infty}^{\infty} y(t)^T x(t)dt.$ |
| $\langle x, y\rangle_\tau$ | $= \langle P_\tau x, P_\tau y\rangle = \int_{-\tau}^{\tau} y(t)^T x(t)dt.$ |
| $\lVert x\rVert_2$ | $= \sqrt{\langle x, x\rangle}.$ |
| $L_2$ | Space of possibly vector valued signals $x$ for which $\lVert x\rVert_2$ exists. |
| $x(t) * y(t)$ | $= \int_{-\infty}^{\infty} x(\tau)y^T(t-\tau)d\tau$, (convolution). |
| $x^*$ | $x^*(t) = x^T(-t)$ if $x$ is a real signal. |
| $r_{xy}(t)$ | $= x * y^* = \int_{-\infty}^{\infty} x(t+\tau)y^T(\tau)d\tau$, (correlation function). |
| $\widehat{x}$ | $= \mathcal{F}x = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$, (Fourier transform). |
| $\lVert x\rVert_1$ | $= \int_{-\infty}^{\infty} \lvert x(t)\rvert dt$ (for scalar valued signals) |
| $l_1$ | Space of signals $x$ for which $\lVert x\rVert_1$ exists. |
| $\mathcal{C}^1$ | Set of continuous once-differentiable mappings. |

**Definition 1** An operator $F : L_2 \longrightarrow L_2$ is said to be *positive* if $\langle x, Fx\rangle \geq 0\ \forall x \in L_2$. If, additionally, there exists a constant $\delta > 0$ such that $\langle x, Fx\rangle \geq \delta\|x\|^2\ \forall x \in L_2$, then $F$ is said to be *strongly positive.*

The constraint (5) is called an *incremental positivity* condition [6]. It is a MIMO generalization of the SISO property of monotonicity.

**Definition 2** $\mathcal{M}_{odd}$ denotes the class of MIMO transfer functions (convolution operators) $M : x \mapsto m * x$ where

$$\widehat{m}(j\omega) \doteq m_0 - \widehat{z}(j\omega)\ \ \forall\omega \ \text{ and } \ m_0 - \|z\|_1 > 0. \quad (9)$$

The subclass obtained under the restriction $z(t) \geq 0\ \forall t$ is designated $\mathcal{M}$. The elements of $\mathcal{M}$ and $\mathcal{M}_{odd}$ are called *Zames-Falb multipliers.*

**Remark 1** Note that $M \in \mathcal{M}$ implies $M$ satisfies (1)–(3). Observe that the Nyquist locus of a Zames-Falb multiplier lies in the open right-half $s$-plane inside a disk centered at $m_0 > 0$ having radius $\|z\|_1 < m_0$. ■

**Definition 3** $\mathcal{N}$ denotes the class of MIMO nonlinearities $N_v : \mathbb{R}^n \to \mathbb{R}^n$ for which (4)–(6) hold and $\mathcal{N}_{odd}$ denotes its subclass

$$\mathcal{N}_{odd} \doteq \{N_v \in \mathcal{N} : N_v(x) = -N_v(-x)\ \forall x \in \mathbb{R}^n\}.$$

In the remainder of the paper, we shall denote the MIMO nonlinearity as simply $N$ rather than $N_v$, for the ease of notation. The following lemma provides a useful MIMO generalization of certain features of Lemmas 7 and 8 of [1].

**Lemma 1** Let $P : \mathbb{R}^n \longrightarrow \mathbb{R}, P \in \mathcal{C}^1$ be a convex function, $x \in \mathbb{R}^n$. Define $y \doteq N(x) \doteq \frac{\partial P(x)}{\partial x}$. Then,

$$\operatorname{trace}[r_{yx}(0) - r_{yx}(\tau)] \geq 0 \qquad \forall \tau \in \mathbb{R}. \tag{10}$$

Furthermore if $N$ is odd, then

$$\operatorname{trace}[r_{yx}(0)] \geq |\operatorname{trace}[r_{yx}(\tau)]| \qquad \forall \tau \in \mathbb{R}. \tag{11}$$

**Proof:**
$P$ convex implies (see [7]) $P(\tilde{x}) \geq P(x) + (\tilde{x} - x)^T P'(x),\ \forall x, \tilde{x} \in \mathbb{R}^n$. In particular,

$$(x(t) - x(t+\tau))^T y(t) \geq P(x(t+\tau)) - P(x(t)). \tag{12}$$

Integrating (12) w.r.t. $t$ over $(-\infty, \infty)$, it follows that $\operatorname{trace}[r_{yx}(0) - r_{yx}(-\tau)] \geq 0$.

If in addition $N$ is odd, then $P$ is an even function so that, in particular, the inequality $(x(t) + x(t+\tau))^T y(t) \geq P(x(t+\tau)) - P(x(t))$ holds along with (12) $\forall t, \tau \in \mathbb{R}$; whence,

$$x^T(t)y(t) + |x^T(t+\tau)y(t)| \geq P(x(t+\tau)) - P(x(t)). \tag{13}$$

Integrating (12) and (13) w.r.t. $t$ over $(-\infty, \infty)$ the results (10) and (11) follow. QED.

## 4 Problem Formulation

The problems of interest are as follows.

**Problem 1** Find the greatest subclass $\tilde{\mathcal{N}} \subset \mathcal{N}$ (or $\tilde{\mathcal{N}}_{odd} \subset \mathcal{N}_{odd}$) whose positivity is preserved by *every* multiplier $M \in \mathcal{M}$ (or, respectively, $M \in \mathcal{M}_{odd}$).

**Problem 2** Find the greatest subclass $\tilde{\mathcal{M}} \subset \mathcal{M}$ (or $\tilde{\mathcal{M}}_{odd} \subset \mathcal{M}_{odd}$) which is positivity preserving for *all* nonlinearities in $\mathcal{N}$ (or, respectively, $\mathcal{N}_{odd}$).

Observe that [1] claimed $\tilde{\mathcal{N}} = \mathcal{N}$ ($\tilde{\mathcal{N}}_{odd} = \mathcal{N}_{odd}$) and, alternatively, $\tilde{\mathcal{M}} = \mathcal{M}$ ($\tilde{\mathcal{M}}_{odd} = \mathcal{M}_{odd}$).

## 5 Main Result

**Theorem 1** Suppose $N \in \mathcal{N}, N \in \mathcal{C}^1$ (or $N \in \mathcal{N}_{odd}, N \in \mathcal{C}^1$). Then, $M^*N$ is positive for all $M \in \mathcal{M}$ (or, respectively, $M \in \mathcal{M}_{odd}$) if and only if,

$$\operatorname{skew}(\frac{\partial N(x)}{\partial x}) = 0\ \forall\ x. \tag{14}$$

**Proof:**
**(if)** Suppose $\operatorname{skew}(\frac{\partial N(x)}{\partial x}) = 0\ \forall\ x$. Define $y = N(x)$. Then,

$$\begin{aligned}\langle M^*N(x), x\rangle &= \langle m_0^* y, x\rangle - \langle z^* * y, x\rangle \qquad (15)\\ &= \left(m_0 - \int_{-\infty}^{\infty} z(t)dt\right) \operatorname{trace}[r_{yx}(0)] + \int_{-\infty}^{\infty} z(t)\epsilon(t)dt\\ &\geq (m_0 - \|z\|_1) \operatorname{trace}[r_{yx}(0)] + \int_{-\infty}^{\infty} z(t)\epsilon(t)dt. \qquad (16)\end{aligned}$$

Since $\operatorname{skew}(\frac{\partial N(x)}{\partial x}) = 0\ \forall x$, Lemma A4 (see Appendix) insures that there exists a continuously differentiable function $P : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $N$ is the gradient of it. Furthermore, condition (5) on $N$ implies that $P$ is convex [7, Chapter 6, pp. 116]. Then, Lemma 1 insures that $\epsilon(t) \doteq \operatorname{trace}[r_{xy}(0) - r_{xy}(t)] \geq 0\ \forall t$, (4)–(5) insure that $\operatorname{trace}[r_{yx}(0)] \geq 0$, and (9) insures that $m_0 - \|z\|_1 > 0$. If $M \in \mathcal{M}$ then $z(t) > 0\ \forall t$ and positivity of $M^*N$ follows from (16). When $M \in \mathcal{M}_{odd}$, using (11), (15) yields

$$\begin{aligned}\langle M^*N(x), x\rangle &= \langle M^*y, x\rangle\\ &\geq \operatorname{trace}[m_0 r_{yx}(0)] - \operatorname{trace}[\int_{-\infty}^{\infty} |z(-t)| r_{yx}(0)dt]\\ &= (m_0 - ||z||_1) \operatorname{trace}[r_{yx}(0)].\end{aligned}$$

Since $m_0 - ||z||_1 > 0$ and $\operatorname{trace}[r_{yx}(0)] \geq 0$, we may conclude that $M^*N$ is positive.

**(only if)** Suppose $\operatorname{skew}(\frac{\partial N}{\partial x}(x_0)) \neq 0$ for some $x_0 \in \mathbb{R}^n$. We shall demonstrate that for $x_\tau(t) \doteq P_\tau(x_0 + \epsilon y(t))$ (where $y(t) \doteq \operatorname{Re}(y_0 e^{j\omega_0 t})$), $\langle M^*Nx_\tau, x_\tau\rangle < 0$ for some $y_0$, all $\tau$ sufficiently large, and all $\epsilon$ sufficiently small. The basic idea of the proof is to choose $M$ from a sequence of Zames-Falb multipliers converging pointwise to $\widehat{m}(j\omega) = j\omega$ so that $M(0) \to 0$, $M(j\omega_0) \to j\omega_0$ and

$$\begin{aligned}&\lim_{\tau\to\infty} \tfrac{1}{\tau} \langle M^*Nx_\tau, x_\tau\rangle\\ &\quad = 2\epsilon^2 y_0^* \left(-j\omega_0 \operatorname{skew}(\frac{\partial N}{\partial x}(x_0))\right) y_0 + O(\epsilon^3),\end{aligned}$$

from which we conclude via Lemma A1 that $M^*N$ is not positive since the Hermitian matrix $\left(-j\omega_0 \operatorname{skew}(\frac{\partial N}{\partial x}(x_0))\right)$ is indefinite. The details follow.

Write $\frac{\partial N}{\partial x}(x_0) = A_h + A_s$ where $A_h \doteq \operatorname{herm}(\frac{\partial N}{\partial x}(x_0))$ and $A_s \doteq \operatorname{skew}(\frac{\partial N}{\partial x}(x_0))$. Observe that $A_s, A_h \in \mathbb{R}^{n\times n}$ with $A_h$ positive semidefinite and $jA_s$ indefinite. Since $N : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $A_s$ has diagonal entries $(A_s)_{ii} = 0\ \forall i$. W.l.o.g. suppose $(A_s)_{12} = -a$ so that $(A_s)_{21} = a$ where $a > 0$. Now, let $x(t) = x_0 + \epsilon y(t)$. Let $y(t) = \operatorname{Re}(y_0 e^{j\omega_0 t})$ where

$$y_0 = [1\ j\ 0\ \dots\ 0]^T. \tag{17}$$

Define $x_\tau \doteq P_\tau x, y_\tau \doteq P_\tau y$. Note that $x_\tau, y_\tau \in L_2$. Consider the $\epsilon$-dependent multiplier $M_\epsilon$ whose Fourier transform is given as

$$\widehat{m}_\epsilon(j\omega) = \frac{j\omega + \epsilon^3}{\epsilon^3 j\omega + 1}, \quad 0 < \epsilon < 1. \tag{18}$$

For this multiplier, $m_0 = \epsilon^{-3}\delta(t), z(t) = (\epsilon^{-6} - 1)e^{-t/\epsilon^3}u(t)$ where $u(t)$ is the unit step function and $m_0 - ||z||_1 = \epsilon^3 > 0$ so that (9) is satisfied. Hence, for all $0 < \epsilon < 1$, $M_\epsilon \in \mathcal{M} \subset \mathcal{M}_{odd}$ is a Zames-Falb multiplier. Now, from the definition of the derivative matrix we have

$$N(x(t)) = N(x_0 + \epsilon y(t)) = N(x_0) + \epsilon \frac{\partial N}{\partial x}(x_0)y(t) + g(t)$$

where $\|g(t)\| = O(\epsilon^2)\|y(t)\|$. Using the fact that $y(t)$ is periodic with period $T \doteq 2\pi/\omega_0$, we obtain

$$\begin{aligned}
&\lim_{\tau\to\infty} \tfrac{1}{\tau} \langle M_\epsilon^* N(x_\tau), x_\tau\rangle_\tau = \lim_{\tau\to\infty} \tfrac{1}{\tau} \langle N(x_\tau), M_\epsilon x_\tau\rangle_\tau \\
&= \lim_{\tau\to\infty} \tfrac{1}{\tau} \langle N(x), M_\epsilon x\rangle_\tau = \tfrac{1}{T} \langle N(x), M_\epsilon x\rangle_T \\
&= \frac{1}{T}\left\langle N(x_0) + \epsilon\frac{\partial N}{\partial x}(x_0)y + g, M_\epsilon x_0 + \epsilon M_\epsilon y\right\rangle_T \\
&= 2\widehat{m}_\epsilon^*(0)x_0^T N(x_0) \\
&\quad +0 \quad \text{(by orthogonality of sinusoids and constants)} \\
&\quad +0 \quad \text{(by orthogonality of sinusoids and constants)} \\
&\quad +\epsilon^2 y_0^* \,\mathrm{herm}(\widehat{m}_\epsilon^*(j\omega_0)(A_h + A_s))y_0 \\
&\quad +O(\epsilon^2)|\widehat{m}_\epsilon(j0)|\ \|x_0\| + O(\epsilon^3)|\widehat{m}_\epsilon(j0)|. \\
&= \epsilon^2 y_0^* \,\mathrm{herm}(\widehat{m}_\epsilon^*(j\omega_0)(A_h + A_s))y_0 + O(\epsilon^3) \qquad (19)
\end{aligned}$$

where the last equality follows since $\widehat{m}_\epsilon(j0) = \epsilon^3$. Now, as $\epsilon \to 0$, $\widehat{m}_\epsilon(j\omega_0) \to j\omega_0$; whence,

$$\begin{aligned}
&\lim_{\epsilon\to 0} y_0^* \,\mathrm{herm}(\widehat{m}_\epsilon^*(j\omega_0)(A_h + A_s))y_0 \\
&= y_0^*(-j\omega_0 A_s)y_0 = -2\omega_0 a < 0
\end{aligned}$$

where the first equality follows by noting that the facts that $A_h$ is Hermitian implies that $jA_h$ is skew and that $A_s$ is skew implies that $jA_s$ is Hermitian. Therefore, for all $\epsilon > 0$ sufficiently small

$$\lim_{\tau\to\infty} \frac{1}{\tau} \langle M_\epsilon^* N(x_\tau), x_\tau\rangle_\tau = -2\epsilon^2\omega_0 a + O(\epsilon^3) < 0. \qquad (20)$$

QED.

**Theorem 2** Suppose $M \in \mathcal{M}$ (or $M \in \mathcal{M}_{odd}$). Then, $M^*N$ is positive for all $N \in \mathcal{N}, N \in \mathcal{C}^1$ (or, respectively, $N \in \mathcal{N}_{odd}, N \in \mathcal{C}^1$) if, and only if,

$$\mathrm{Im}(\widehat{m}(j\omega)) = 0 \text{ for all } \omega. \qquad (21)$$

**Proof:**
**(if)** First, consider the case $M \in \mathcal{M}, N \in \mathcal{N}, N \in \mathcal{C}^1$. Suppose that (21) holds. It follows from (21) and the definition of the Fourier transform that $m(t) \doteq m_0 - z(t)$ must be an even function; i.e., $z(t) = z(-t)$, $\forall t$. Thus, for all $y \doteq Nx$,

$$\begin{aligned}
&\langle M^* N(x), x\rangle = \langle M^* y, x\rangle = \langle m_0^* y, x\rangle - \langle z^* * y, x\rangle \\
&= \mathrm{trace}[m_0 r_{yx}(0)] - \mathrm{trace}[\int_{-\infty}^{\infty} z(-t) r_{yx}(-t)dt] \qquad (22) \\
&= \mathrm{trace}[m_0 r_{yx}(0)] - \\
&\quad \mathrm{trace}[\int_{-\infty}^{\infty} 1/2(z(-t) + z(t))r_{yx}(-t)dt] \qquad (23) \\
&= \mathrm{trace}\left[\left(m_0 - \int_{-\infty}^{\infty} z(t)dt\right) r_{yx}(0)\right] \\
&\quad + \frac{1}{2}\mathrm{trace}\left[\int_{-\infty}^{\infty} z(t)\left(2r_{yx}(0) - r_{yx}(t) - r_{yx}(-t)\right)dt\right] \\
&\geq 0.
\end{aligned}$$

(23) follows from (22) since $z(t) = z(-t)$). Since $m_0 - \int_{-\infty}^{\infty} z(t)dt \geq 0$, the last inequality follows from (9) and Lemma A3.

When $M \in \mathcal{M}_{odd}, N \in \mathcal{N}_{odd}$ and $N \in \mathcal{C}^1$, the result can be seen to hold by using arguments similar to the ones employed for that case in the proof of the (if) part of Theorem 1 in conjunction with (24).

**(only if)** Noting that $\mathcal{N}_{odd} \subset \mathcal{N}$, it suffices to find an instance of an $N \in \mathcal{N}_{odd}$ for which $M^*N$ is not positive whenever $\mathrm{Im}(\widehat{m}(j\omega)) \neq 0$. The counterexample given in Section 2 is one such instance. QED.

**Remark 2** Multiplier characterization provided by Theorem 2 does not reduce conservatism in stability analysis if the system in Fig. 1 has a *full-block* feedback nonlinearity $N$ since there is no distinction between positivity of the feedforward element $H$ and that of $MH$. However, if $N$ is *block-diagonal*, say $N \doteq \mathrm{diag}(N_1, N_2, \ldots, N_m)$ where $N_i : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}^{n_i}$ satisfy (4)–(6), then Theorem 2 can be seen to be useful as follows. Every multiplier, say $M_i$, which preserves positivity of $N_i$ has a (spectral) factorization of the form (see, e.g., [1, Lemma 3]) $M_i = M_{-i}M_{+i} = D_i^* D_i$ where $D_i = M_{+i} = M_{-i}^*$, and both $D_i$ and $D_i^{-1}$ are stable. Observe that positivity of $N$ is preserved by block-diagonal scaling $N \leftarrow DND^{-1}$ where the frequency-response matrix $\widehat{d}(j\omega)$ of $D$ has the form $\widehat{d}(j\omega) \doteq \mathrm{diag}(\widehat{d}_1(j\omega)I_1, \widehat{d}_2(j\omega)I_2, \ldots, \widehat{d}_n(j\omega)I_n)$ where $I_i$ denote $n_i \times n_i$ identity matrices and the scalars $\widehat{d}_i(j\omega)$ denote the frequency-responses of the spectral factors $D_i$. That is, for purposes of stability analysis via the equivalent system in Fig. 2, we may take $M_-^* = M_+ = D$. Stability is then assured $MH \equiv D^*DH$ is positive or, equivalently, if the diagonally-scaled linear operator. Note that *in this case*, positivity of $MH$ is *not* equivalent to that of $H$, which means that the real multipliers provided by Theorem 2 can reduce conservativeness in deciding stability of the system. ■

## 6 Discussion

Theorem 1 and Theorem 2 completely characterize the solution to Problem 1 and Problem 2, respectively. The reader will doubtless realize that the case of multiple scalar feedback nonlinearities, repeated or otherwise, which has been covered in the literature to date, implicitly assumes that the feedback nonlinearity is the gradient of a potential function. We believe that our Theorem 1 and Theorem 2 can be used with the results for the case of repeated scalar nonlinearities derived by D'Amato et al. [4] to derive those for the case of repeated block diagonal nonlinearities. A practical implementation of the Zames-Falb multipliers to solve the stability problem in the SISO setting amounts to solving an infinite dimensional linear program (see [8]) which has been shown (see [9]) to require typically less than 10 free elements owing to convexity of the problem. We speculate that a practical implementation in MIMO setting would lead to a linear matrix inequality problem which can be solved efficiently.

## 7 Conclusion

In their celebrated 1968 paper on nonlinear stability, Zames and Falb [1] determined a class of multipliers that preserve the incremental-positivity of norm-bounded, time-invariant, memoryless SISO nonlinearities. Results therein were proved only for the SISO nonlinearities and it was conjectured that the results might also be generalized to hold in the MIMO case. Validity of this conjecture has remained suspect, as is evident from the comments of Willems [5, Chapter 3, pp. 66] as well as from the scarcity of literature on the analysis of such systems in MIMO setting even though the case of multiple scalar nonlinearities has received some attention (see [2], [3], [4] and references therein). In this paper, we have demonstrated that their conjecture, as it stands, is incorrect and that for it to hold true, a further restriction needs to be applied. Specifically, it suffices either to restrict the MIMO nonlinearity to be the gradient of a convex real-valued function or to restrict the multiplier to be a real-valued function of frequency.

## Appendix: Background Results

**Lemma A1** Let $G : L_2 \to L_2$ be a linear time-invariant operator having $n \times n$ frequency response matrix $\widehat{g}(j\omega)$. Then,

$$\langle Gx, x\rangle \;\geq 0 \;\; \forall x \in L_2 \Longleftrightarrow \lambda_{\min}(\text{herm}(\widehat{g}(j\omega))) \geq 0 \;\; \forall \omega.$$

**Lemma A2** Let $N : \mathbb{R}^n \longrightarrow \mathbb{R}^n, N \in \mathcal{C}^1$. Then,

$$(x-y)^T N(x) + (y-x)^T N(y) \geq 0 \;\; \forall x, y \in \mathbb{R}^n \quad \text{(A1)}$$

if, and only if,

$$\text{herm}(\frac{\partial N(x)}{\partial x}) \geq 0 \;\; \forall x \in \mathbb{R}^n. \qquad \text{(A2)}$$

**Lemma A3** Let $N : \mathbb{R}^n \longrightarrow \mathbb{R}^n, N \in \mathcal{C}^1$ and $y = N(x)$. If $\text{herm}(\frac{\partial N(x)}{\partial x}) \geq 0 \; \forall x \in \mathbb{R}^n$ then

$$\text{trace}[2r_{yx}(0) - r_{yx}(\tau) - r_{yx}(-\tau)] \geq 0 \; \forall \tau \in \mathbb{R}.$$

**Lemma A4** [10, Chapter 5, pp. 359]
Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \; f \in \mathcal{C}^1, \; x \in \mathbb{R}^n$. Then,

$$P(x) \doteq \int_0^x f(x)dx \qquad \text{(A3)}$$

exists and is path independent if, and only if,

$$\text{skew}(\frac{\partial f(x)}{\partial x}) = 0 \;\; \forall x \in \mathbb{R}^n. \qquad \text{(A4)}$$

**Remark 3** Given a continuously differentiable vector field $f$, the above lemma gives a necessary and sufficiency condition for the existence of a potential function $P$. The condition (A4) generalizes the familiar condition $\text{curl}(f) = 0$ satisfied by electric fields $f : \mathbb{R}^3 \to \mathbb{R}^3$ as a consequence of Maxwell's electromagnetic field equations [10]. ∎

## References


[1] G. Zames and P. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM J. Control and Optimization*, 6:89–108, 1968.

[2] W. Haddad and V. Kapila. Absolute stability criteria for multiple slope-restricted monotonic nonlinearities. *IEEE Trans. Autom. Control*, AC-40(2):361–365, February 1995.

[3] J. Suykens, J. Vandewalle, and B. De Moor. An absolute stability criterion for the Lur'e problem with sector and slope restricted nonlinearities. *IEEE Trans. Circuits and Systems — I: Fundamental Theory and Applications*, CAS-45(9):1007–1009, September 1998.

[4] F. D'Amato, A. Megretski, U. Jönsson, and M. Rotea. Integral quadratic constraints for monotonic and slope-restricted diagonal operators. In *Proc. American Control Conf.*, pages 2375–2379, San Diego, CA, June 2–4, 1999. IEEE Press, New York.

[5] J. C. Willems. Least square stationary optimal control and the algebraic Riccati equation. *IEEE Trans. Autom. Control*, AC-16:621–627, 1971.

[6] G. Zames. On the input–output stability of time-varying nonlinear feedback systems — Part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Trans. Autom. Control*, AC-11(2):228–238, April 1966.

[7] D. Luenberger. *Introduction to Linear and Nonlinear Programming.* Addison-Wesley, Reading, MA, 1973.

[8] M. G. Safonov and G. Wyetzner. Computer-aided stability analysis renders Popov criterion obsolete. *IEEE Trans. Autom. Control*, AC-32(12):1128–1131, 1987.

[9] P. B. Gapski and J. C. Geromel. A convex approach to the absolute stability problem. *IEEE Trans. Autom. Control*, AC-39(9):1929–1932, September 1994.

[10] R. Williamson, R. Crowell, and H. Trotter. *Calculus of Vector Functions (2nd Ed.).* Prentice Hall, Englewood-Cliffs, NJ, 1968.