Multiplier IQCs for Uncertain Time-delays[1]

Myungsoo Jun and Michael G. Safonov[2]
Dept. of Electrical Engineering — Systems
University of Southern California
Los Angeles, CA 90089-2563, USA

Key Words: Time-delay system, stability criteria, multiplier, robust control, IQC

Abstract

This paper describes a set of delay-dependent IQC's for time-delay uncertainty. The set is linearly parameterized in terms of the frequency-response of a complex scalar-valued multiplier. Using LMI optimization techniques, one may compute optimal multipliers and thereby obtain less conservative IQC stability robustness bounds for systems with uncertain time-delays.

1 Introduction

Stability criteria for time-delay systems tend to fall into one of two categories according to their dependence upon delay size: *delay-dependent* or *delay-independent*. Delay-independent criteria provide conditions for stability without regard for the size of the time delays. They tend to be more conservative than delay-dependent criteria which may exploit prior knowledge of upper-bounds on the amount of time-delay.

The robust stability methodology is useful in dealing with structured uncertainties (see [2],[4]). Time-delays can be considered as structured uncertainties and time-delay systems can be analyzed using these robust control theories[11]. Many of methods that have been developed within the area of robust control during the last decade have been shown to be reformulated to fall within the framework of the integral quadratic constraints (IQC's)[8]. Fu *et al.*[5] provided two delay-dependent results for robust stability using this IQC approach and the linear matrix inequalities (LMI's) technique, which give an estimate of the maximum time-delay which preserves robust stability.

Some recent papers on time-delay systems, like [9], [3] and [7], derive sufficient conditions for stability in the form of LMI using Lyapunov functionals. Others have used input-output stability theories such as the small-gain criterion [12] and its generalizations, like the IQC stability analysis method [8]. The input-output methods seem to offer advantages over Lyapunov methods in facilitating the decomposition of the stability robustness analysis problem for complex systems having several sorts of uncertain subsystems into subproblems of finding an IQC for each of the subsystems — e.g., for each uncertain time-delay and each nonlinearity or other uncertainty. Once the subsystem IQC's are in hand, stability analysis for the composite system is then a relatively straightforward matter of optimizing IQC scalings (and, sometimes, other free parameters like Popov multipliers) in an effort to identify a single aggregate IQC for the system.

This paper considers robust stability analysis of systems with time delay based on an IQC approach. It is organized as follows: Notation and preliminary background are provided in Section 2. The problem formulation is given in Section 3. The main result is derived in Section 4 where it is shown that the class of known delay-dependent IQC's for time delays can be generalized to a larger linearly-parameterized class. Discussion and comparison with other results are in Section 6. Finally, results are summarized and conclusions are stated in Section 7.

2 Background and Preliminaries

Definition 1 *(cf. [8]) Consider the feedback system in Figure 1 where G, Δ are causal operators and G has transfer function $G(s)$. We say that the interconnection G and Δ is* well-posed *if the operator* $\begin{bmatrix} I & -G \\ -\Delta & I \end{bmatrix}$ *has a causal inverse. The interconnection is* stable *if, additionally, the inverse is bounded.*

Theorem 1 (The IQC Theorem) *[8, 6] Let $G(s) \in \mathcal{RH}_\infty^{l\times m}$, and let $\Delta : \mathcal{L}_{2e}^{l}[0,\infty) \mapsto \mathcal{L}_{2e}^{m}[0,\infty)$ be a bounded causal operator. Assume that:*

i) for every $\alpha \in [0,1]$, the interconnection of G and Δ_α is well-posed where Δ_α is a parameterization of Δ which satisfies

a) $\Delta = \Delta_\alpha|_{\alpha=1}$,

[1] Research sponsored in part by AFOSR Grant F49620-98-1-0026

[2] Corresponding author; email msafonov@usc.edu; fax +1-213-740-4449.

Table 1: Notation

Symbol	Meaning
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}_+$	Set of positive real numbers
$\mathbb{C}$	Set of all complex numbers
$A(s)^*$	$A(-s)^T$, conjugate transpose
$\mathrm{herm}(m)$	$= \frac{1}{2}(m + m^*)$
$\mathrm{skew}(m)$	$= \frac{1}{2}(m - m^*)$
$\mathfrak{Re}(\cdot)$	Real part of $(\cdot)$
$\mathfrak{Im}(\cdot)$	Imaginary part of $(\cdot)$
$\hat{x}(j\omega)$	Fourier transform of the signal $x(t)$
$< x, y >$	$= \int_{-\infty}^{\infty} y(t)^T x(t) dt$
	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(j\omega)^* \hat{x}(j\omega) d\omega$
$\|x\|$	$= \sqrt{< x, x >}$

b) Δ_α is bounded and causal for $\alpha \in [0, 1]$,

c) there exists $\gamma > 0$ such that

$$\|\Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y)\| \leq \gamma |\alpha_1 - \alpha_2| \cdot \|y\| \quad (1)$$

for all $\alpha_1, \alpha_2 \in [0, 1]$,

ii) the interconnection of G and $\Delta_\alpha|_{\alpha=0}$ is stable,

iii) for every $\alpha \in [0, 1]$, the IQC defined by Π is satisfied by Δ_α, that is,

$$\left\langle \Pi \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix}, \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix} \right\rangle \geq 0, \quad (2)$$

iv) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \forall \omega \in \mathbb{R}. \quad (3)$$

Then, the feedback interconnection of G and Δ is stable.

The values $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ of $\Pi(j\omega)$ represent the small gain theorem and the positivity theorem. The positivity theorem with multiplier can be reformulated with IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & 0 \end{bmatrix}. \quad (4)$$

3 Problem Formulation

Consider a time-delay uncertainty $\Delta(s) = e^{-\tau s}$ where $\tau \in [0, \bar{\tau})$ and τ is assumed to be constant but unknown. In this paper, we provide a class of Π's for time delay uncertainty with an appropriate multiplier $M(j\omega)$ and transformation $S(j\omega)$.

Problem 1 *Given a time-delay uncertainty $\Delta(s) = e^{-\tau s}$, find a class of Π which satisfies the IQC (2).*

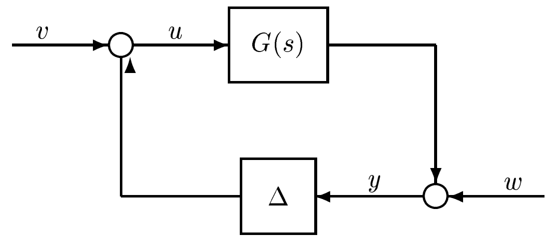

Figure 1: Basic feedback configuration

4 Main Results

Lemma 1 *Suppose $\Pi_{ii} \in \mathbb{R}, i = 1, 2$ and $\Pi_{12} \in \mathbb{C}$. If $\det \Pi < 0$ and $\Pi_{22} \neq 0$, then the locus $\Delta(j\omega)$ which satisfy the quadratic equation*

$$\begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} = 0 \quad (5)$$

is the circle $\mathcal{C}(\Pi(j\omega)) = \{ \Delta \mid |\Delta(j\omega) - c| = r \}$ where $c = -\frac{\Pi_{12}^*}{\Pi_{22}}$ and $r = \sqrt{\frac{|\Pi_{12}|^2}{\Pi_{22}^2} - \frac{\Pi_{11}}{\Pi_{22}}} = \frac{\sqrt{-\det \Pi}}{|\Pi_{22}|}$.

Proof:

$$\begin{aligned} 0 &= \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} \\ &= \Pi_{11} + \Pi_{12}\Delta + \Delta^*\Pi_{12}^* + \Delta^*\Pi_{22}\Delta \\ &= \Pi_{22}\left(\Delta^*\Delta + \Pi_{22}^{-1}\Delta^*\Pi_{12}^* + \Pi_{22}^{-1}\Pi_{12}\Delta\right) + \Pi_{11} \\ &= \Pi_{22}\left(\Delta + \Pi_{22}^{-1}\Pi_{12}^*\right)^*\left(\Delta + \Pi_{22}^{-1}\Pi_{12}^*\right) + \\ &\quad \Pi_{11} - \Pi_{12}\Pi_{12}^*\Pi_{22}^{-1} \\ &= \Pi_{22}|\Delta + \Pi_{22}^{-1}\Pi_{12}^*|^2 + \Pi_{11} - \Pi_{12}\Pi_{12}^*\Pi_{22}^{-1} \\ &= \Pi_{22}\left(|\Delta - c|^2 - r^2\right) \end{aligned}$$

■

If $\Pi_{22} = 0$ and $\Pi_{12} \neq 0$, the set of $\Delta(j\omega)$'s which satisfy Eq. (5) is the line which can be represented by the equation

$$\begin{aligned} &(\Pi_{12} + \Pi_{12}^*) \cdot \frac{\Delta(j\omega) + \Delta(j\omega)^*}{2} + \\ &\quad (\Pi_{12} - \Pi_{12}^*) \cdot \frac{\Delta(j\omega) - \Delta(j\omega)^*}{2} + \Pi_{11} = 0, \end{aligned} \quad (6)$$

that is,

$$\begin{aligned} &2\mathfrak{Re}(\Pi_{12}) \cdot \mathfrak{Re}(\Delta(j\omega)) - 2\mathfrak{Im}(\Pi_{12}) \cdot \mathfrak{Im}(\Delta(j\omega)) \\ &\quad + \Pi_{11} = 0. \end{aligned} \quad (7)$$

We define $\tilde{y}(j\omega)$ and $\tilde{u}(j\omega)$ to satisfy the relation $\begin{bmatrix} \tilde{y} \\ \tilde{u} \end{bmatrix} = S(j\omega) \begin{bmatrix} y \\ u \end{bmatrix}$ with appropriate transformation matrix $S(j\omega)$. We also define $\Delta(j\omega) \triangleq u(j\omega)y(j\omega)^{-1}$ and

$\tilde{\Delta}(j\omega) \triangleq \tilde{u}(j\omega)\tilde{y}(j\omega)^{-1}$. We will use the symbol ω_* to denote $\omega_* \triangleq \frac{\bar{\tau}\omega}{2}$.

The Lemma 2 talks about a transformation $S(j\omega)$ which maps an arc to the positive real line.

Lemma 2 *Let the transformation matrix $S(j\omega)$ be*

$$S(j\omega) \triangleq \begin{bmatrix} -\cos\omega_* + j\sin\omega_* & \cos\omega_* + j\sin\omega_* \\ \sin\omega_* & -\sin\omega_* \end{bmatrix}. \quad (8)$$

Then, $\Delta(j\omega) = \{\ \Delta \mid \ \Delta = e^{-j\varphi},\ \varphi \in [0, 2\omega_*)\ \}$ and $\omega_* \in [0,\pi)$ if and only if $\tilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$.

Proof: *(only if)*

$$\begin{aligned}
\tilde{\Delta}(j\omega) &= \frac{\sin\omega_* - \sin\omega_*\Delta(j\omega)}{(-\cos\omega_* + j\sin\omega_*) + (\cos\omega_* + j\sin\omega_*)\Delta(j\omega)} \\
&= \frac{1}{-\cot\omega_* + \dfrac{j + j\Delta(j\omega)}{1-\Delta(j\omega)}} \\
&= \frac{1}{-\cot\omega_* + \dfrac{\sin\varphi + j(1+\cos\varphi)}{1-\cos\varphi + j\sin\varphi}} \\
&= \frac{1}{-\cot\omega_* + \dfrac{\sin\varphi}{1-\cos\varphi}} \\
&= \frac{1}{-\cot\omega_* + \dfrac{1}{\tan\frac{\varphi}{2}}} \\
&= \frac{1}{-\cot\omega_* + \cot\frac{\varphi}{2}}.
\end{aligned}$$

We can see that $\tilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$ since $\varphi/2 < \omega_*$ and $-\cot\omega_*$ is strictly increasing when $\omega_* \in [0,\pi)$.

(if) This can be easily proved from the fact that $S(j\omega)$ is invertible for $\forall\ \omega_* \in [0,\pi)$ and $\tilde{\Delta}(j\omega)$ has all values in $\mathbb{R}_+ \cup \{0\}$. ■

Now we state our main theorem which provides a linearly parameterized set of Π's for time delay uncertainty.

Theorem 2 (Main Theorem) *Let $\Delta(j\omega) = e^{-j\omega\tau}$, $\tau \in [0,\bar{\tau})$, $\omega_* \triangleq \frac{\bar{\tau}\omega}{2}$. Then, $\Delta(j\omega)$ satisfies the IQC (2) for*

$$\Pi(j\omega) = \begin{cases} herm\left(M(j\omega)\begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}-e^{-j\omega_*} & e^{j\omega_*}\end{bmatrix}\right), & \omega_* \in [0,\pi) \\ herm\left(M(j\omega)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right), & otherwise \end{cases} \quad (9)$$

with $\mathfrak{Re}(M(j\omega)) \geq 0$.

Proof: When $\omega_* \in [0,\pi)$, $\Delta(j\omega)$ is represented as

$$\Delta(j\omega) = \{\ \Delta \mid \ \Delta = e^{-j\varphi},\ \varphi \in [0,2\omega_*)\ \}. \quad (10)$$

And $\Pi(j\omega)$ can be said to be

$$\Pi(j\omega) = S(j\omega)^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} S(j\omega) \quad (11)$$

where $S(j\omega)$ is the same transformation matrix as in Eq. (8). Then, we can say that

$$\begin{bmatrix} 1 \\ \tilde{\Delta}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \tilde{\Delta}(j\omega) \end{bmatrix} \geq 0 \quad (12)$$

for $\omega_* \in [0,\pi)$ since $\tilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$ by Lemma 2 and $\mathfrak{Re}(M(j\omega)) \geq 0$. This is equivalent to saying that

$$\begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix} \geq 0 \quad (13)$$

for $\omega_* \in [0,\pi)$ by the fact $\begin{bmatrix}\tilde{y}\\ \tilde{u}\end{bmatrix} = S(j\omega)\begin{bmatrix}y\\u\end{bmatrix}$.

When $\omega_* \notin [0,\pi)$, the $\Pi(j\omega)$

$$\Pi(j\omega) = \text{herm}\left\{ M(j\omega) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \quad (14)$$

represents a circle with center at the origin and radius 1 by Lemma 1. And since $\Delta(j\omega)$ for $\omega_* \notin [0,\pi)$ is also a circle with center at the origin and radius 1, it can be said that

$$\begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} M(j\omega) & 0 \\ 0 & -M(j\omega) \end{bmatrix} \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} = 0 \quad (15)$$

for $\mathfrak{Re}(M(j\omega)) \geq 0$.

Thus, this proves that the $\Pi(j\omega)$ in Eq. (9) satisfies the IQC (2) for $\forall\,\omega \in \mathbb{R}$. ■

5 Parameterization of Δ_α

We can see that the IQC (2) does not hold for all $\alpha \in [0,1]$ if we use a simple *linear* parameterization of Δ, that is, $\Delta_\alpha = \alpha\Delta$. So, we have to find other parameterizations than linear parameterization in order to satisfy the condition a), b) and c) in Theorem 1.

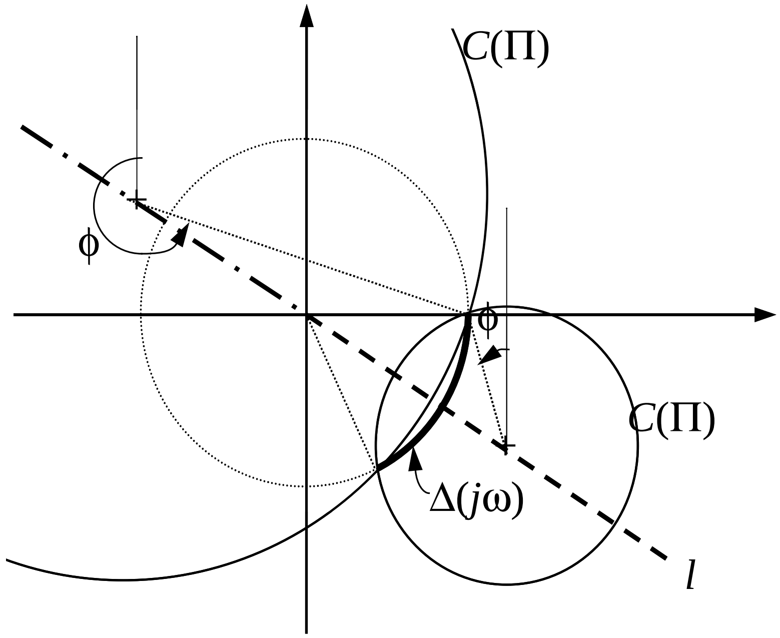

Figure 2: The graph of $\Delta(j\omega)$ and $\mathcal{C}(\Pi(j\omega))$ when $\omega_* \in [0,\pi)$. $\mathcal{C}(\Pi)$ is the circle with center on the line ℓ. The phase angle $\phi \triangleq \angle M(j\omega)$ determines which circle.

Theorem 3 *The parameterization of* $\Delta(j\omega)$ *defined by*

$$\Delta_\alpha(j\omega) = e^{-j\omega\tau\alpha} \tag{16}$$

satisfies the IQC (2) for all $\alpha \in [0,1]$ *and conditions a),b) and c) in Theorem 1.*

Proof: It is easy to check that the conditions a) and b) hold with the parameterization (16). Let us check the condition c).

If $\alpha_1 = \alpha_2$, γ can have any value to satisfy the condition c). Now let us prove the case when $\alpha_1 \neq \alpha_2$.

$$\begin{aligned}
&\|\Delta_{\alpha_1} - \Delta_{\alpha_2}\| \\
&\quad = \|e^{-j\omega\tau\alpha_1} - e^{-j\omega\tau\alpha_2}\| \\
&\quad \le \|e^{-j\omega\tau}\| \cdot \|e^{\alpha_1} - e^{\alpha_2}\| \\
&\quad = \|e^{\alpha_1} - e^{\alpha_2}\| \\
&\quad = \left\| \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2} \cdot (\alpha_1 - \alpha_2) \right\| \\
&\quad \le \left\| \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2} \right\| \cdot \ \|\alpha_1 - \alpha_2\| \\
&\quad \le e \cdot |\alpha_1 - \alpha_2|
\end{aligned}$$

since $0 < \frac{e^{\alpha_1} - e^{\alpha_2}}{\alpha_1 - \alpha_2} \le e = 2.7182\cdots$ for $\alpha_1, \alpha_2 \in [0,1]$. Thus, if we set $\gamma = e$, we can see that the condition c) is satisfied with this value of γ. ∎

6 Discussion

It can be shown that the IQC's for time-delays introduced by Megretski *et al.* [8] and by Scorletti [11] are special cases of the more parameterized class of IQC's given by our Theorem 2. In this section, we show that each of these special cases corresponds to a particular choice for our multiplier parameter $M(j\omega)$.

The inequality used by Megretski *et al.* [8] to find an IQC for time delay $\tau \in [0, \bar{\tau}]$ is

$$\begin{aligned}
&\psi_1(\omega_*)(|j\omega_* u(j\omega) + y(j\omega)|^2 - (1+\omega_*^2)|y(j\omega)|^2) \\
&\qquad \ge \psi_2(\omega_*)|y(j\omega) - u(j\omega)|^2)
\end{aligned} \tag{17}$$

where $\psi_{1,2}$ are the functions defined by

$$\psi_1(\omega) = \begin{cases} \frac{\sin\omega}{\omega}, & |\omega| \le \pi \\ 0, & |\omega| > \pi \end{cases},$$

$$\psi_2(\omega) = \begin{cases} \cos\omega, & |\omega| \le \pi \\ 0, & |\omega| > \pi \end{cases}.$$

By Lemma 1 this IQC corresponds to a circle with its center at $\frac{1}{m}(\cos\omega_* - j\sin\omega_*)$ and radius $\sqrt{1/m^2 - (\omega_*\sin\omega_* + \cos\omega_*)/m}$ where $m = \cos\omega_* - \omega_*\sin\omega_*$. If $m > 0$, Eq. (17) is the inside of the circle and otherwise, it is the outside of the circle. Thus by Lemma 1, Megretski *et al.*'s IQC for the time delay is equivalent to the IQC that arises from

$$\Pi = \begin{bmatrix} -\cos\omega_* - \omega_*\sin\omega_* & \cos\omega_* + j\sin\omega_* \\ \cos\omega_* - j\sin\omega_* & -\cos\omega_* + \omega_*\sin\omega_* \end{bmatrix} \tag{18}$$

Comparing the Eq. (18) with the Eq. (9), we can see that we have the Eq. (18) when $M(j\omega) = \sin\omega_* + j\omega_*\sin\omega_*$.

Now let us consider the IQC in [11]. The uncertainty for time delay in [11] is defined as $\Delta(j\omega) = e^{-j\omega\tau} - 1$. Thus, if we reformulate the IQC in [11] with time uncertainty $\Delta(j\omega) = e^{-j\omega\tau}$, then we have

$$\Pi = \begin{bmatrix} -2\cot\omega_* & \cot\omega_* + j \\ \cot\omega_* - j & 0 \end{bmatrix} \tag{19}$$

We can easily see that we can get Eq. (19) from Eq. (9) with $M(j\omega) = 1 + j\cot\omega_*$.

Thus, our result includes the results by [8] and [11]. Furthermore, the Eq. (9) is linear with respect to $M(j\omega)$. This ensures that the problem of finding the optimal multiplier $M(j\omega)$ which provides least conservative bound of delay margin is, at each frequency ω, a linear matrix inequality (LMI); consequently, the optimal multiplier $M(j\omega)$ may be readily computed — see, for example, Boyd *et al.* [1] and Safonov *et al.* [10].

By Lemma 1, it is possible to interpret our multiplier-based IQC's for time-delays in terms of a circle that passes through the two end points of the arc $\Delta(j\omega) = \{\ \Delta \mid \ \Delta = e^{-j\varphi},\ \varphi \in [0, 2\omega_*)\ \}$, 1 and $e^{-2j\omega_*}$, in the Nyquist plane. The phase angle $\phi \triangleq \angle M(j\omega)$ determines which one of the many circles passing through these two points. See Figure 2. Notice that the centers of circles lie on the line ℓ which passes the origin and slope ω_* regardless of the value of $M(j\omega)$. And we can see that the inequality (2) implies the inside or outside of a circle according to the sign of Π_{22}. When $\Pi_{22} < 0$ it is the inside of a circle by Lemma 1 and outside when $\Pi_{22} > 0$. If the center of a circle lies on the dashed ray in Figure 2, the region which the inequality (2) represents is the inside of the circle. On the other hand when the center lies on the dash-dotted ray, it is the outside. Just one of the many possible circles determined by our multipliers was considered in [8] and [11]. A line can be thought as a circle with infinite radius which occurs when $\phi = \pi/2 - \omega_*$. Our multiplier IQC's for time-delay use the multiplier $M(j\omega)$ to parameterize the entire class of *all* circles which pass through those two points.

7 Conclusion

Working from the IQC perspective, we provide a generalized class of $\Pi(j\omega)$'s for time delays which are linearly parameterized in terms a scalar-valued complex multiplier $M(j\omega)$ with $\mathfrak{Re}(M(j\omega)) > 0$. The multipliers may be readily optimized for each specific application using LMI techniques. The results are less conservative than if a particular multiplier were to be prespecified. In particular, we have shown that the time-delay IQC's of Megretski *et al.* [8] and of Scorletti [11] correspond to two such particular choices for the multiplier, which means that our multiplier-parameterized IQC's will generally produce better results.

References

[1] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, Philadelphia, PA, 1994.

[2] J. C. Doyle. Analysis of feedback systems with structured uncertainties. In *IEE Proc.*, volume 129-D(6), pages 242–250, November 1982.

[3] S. H. Esfahani and I. R. Petersen. An LMI approach to the output-feedback guaranteed cost control for uncertain time-delay systems. In *Proc. of IEEE Conf. Decision and Control*, pages 1358–1363, Tampa, FL, December 1998.

[4] M. K. H. Fan, A. L. Tits, and J. C. Doyle. Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. on Auto. Control*, AC-36(1):25–38, January 1991.

[5] M. Fu, H. Li, and S. I. Niculescu. *Robust stability and stabilization of time-delay systems via integral quadratic constraint approach*, volume 228 of *Lecture Notes in Control and Information Science*, chapter 4. Springer, 1998.

[6] U. Jönsson and A. Megretski. The Zames-Falb IQC for critically stable systems. In *Proc. of the American Control Conference*, pages 3612–3616, Philadelphia, PA, 1998.

[7] K. H. Lee, Y. S. Moon, and W. H. Kwon. Robust stability analysis of parametric uncertain time-delay systems. In *Proc. of IEEE Conf. Decision and Control*, pages 1346–1352, Tampa, FL, December 1998.

[8] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. on Auto. Control*, AC-42(6):819–830, June 1997.

[9] S. Phoojaruenchanachai and K. Furuta. Memoryless stabilization of uncertain linear systems including time-varying states delays. *IEEE Trans. on Auto. Control*, AC-37(7):1022–1026, July 1992.

[10] M. G. Safonov and R. Y.Chiang. *Real/complex k_m-synthesis without curve fitting*, volume 56 of *Control and Dynamic Systems*, pages 303–324. Academic Press, New York, NY, 1993.

[11] G. Scorletti. Robustness analysis with time-delay. In *Proc. of IEEE Conf. on Decision and Control*, pages 3824–3829, San Diego, CA, December 1997.

[12] G. Zames. On the input-output stability of time-varying nonlinear feedback systems–Parts I & II: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Trans. on Auto. Control*, AC-11(2 & 3):228–238 & 465–476, April & July 1966.