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Stability Analysis of a System with Time-delayed States [1]

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Abstract

In this paper, we provide a sufficient condition for asymptotic stability of a system with a single time delay in states expressed as a linear matrix inequality(LMI). We formulate the time delay into a inverse uncertainty configuration.

1 Introduction

There have been several studies about stability criteria for time-delay systems. These criteria can be classified into two categories according to their dependence upon delay size : *delay-dependent* or *delay-independent*. Delay-independent criteria provide conditions for stability regardless of size of time delays. Thus, they tend to be more conservative than delay-dependent criteria. Delay-dependent conditions are dependent upon the size of time delays and can give information on the delay margin. However, these conditions could provide much conservative result if a system is stable for any time delay.

Fu *et al.*[2] provided two delay-dependent results for robust stability using the integral quadratic constraints (IQCs) approach and the linear matrix inequalities (LMIs) technique. Kolmanovskii *et al.*[4] gave a mixed delay-dependent/delay-independent condition for linear systems with delayed states. Scorletti[9] proposed an extension of the μ analysis to address the analysis of systems with non-rational uncertainties in a connected set and obtained convex sufficient conditions involving linear matrix inequalities. And many of recent papers on time-delay systems, like [1] and [5], derived sufficient conditions for stability in the form of LMI using Lyapunov functionals.

Most of researches on time-delay systems expressed time-delay uncertainties as simple multiplicative uncertainties, viz., either $\Delta(j\omega) = e^{-j\omega\tau}$ or $\Delta(j\omega) = e^{-j\omega\tau} - 1$. In this paper, we formulate the time-delay uncertainty as $\Delta(j\omega) = \frac{1}{j\omega\tau}(e^{-j\omega\tau} - 1)$. The advantage of this representation is that $\Delta(j\omega)$ can be said to be strictly proper, so it can be used when it is required for the $\Delta(j\omega)$ to be strictly proper. We propose a sufficient condition for asymptotic stability of a system with single time-delay in states which is expressed as an LMI. We transform a frequency dependent matrix inequality that came from IQC theorem to an equivalent non-frequency dependent LMI via Kalman-Yakubovich-Popov Lemma.

This paper is organized as follows : The problem formulation is given in Section 2. Notation and preliminary background are described in Section 3. Our main result is provided in Section 4. Numerical example and discussion are given in Section 5 and Section 6. Finally, conclusions are stated in Section 7.

2 Problem Formulation

Consider an uncertain linear time invariant system with single time-delay in a state

$$\dot{x}(t) = A_0 x(t) + A_d x(t-\tau) \tag{1}$$

where $\tau \in [0, \bar{\tau}]$ and $A_0 + A_d \in \mathbb{R}^{n \times n}$ is Hurwitz, that is, the system is stable if there is no time delay. τ is assumed to be constant but unknown.

Problem 1 *Given a system (1), find a delay τ_{max} which maintains the system (1) asymptotically stable for any positive τ which is smaller than τ_{max}.*

3 Preliminaries

Definition 1 *(cf. [6]) Consider the feedback system in Figure 1 where G, Δ are causal operators and G has transfer function $G(s)$. We say that the interconnection G and Δ is* well-posed *if the operator* $\begin{bmatrix} I & -G \\ -\Delta & I \end{bmatrix}$

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Table 1: Notation

Symbol	Meaning
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}_+$	Set of positive real numbers
$\mathbb{C}$	Set of all complex numbers
$\mathbb{C}_+$	Set of all complex numbers with positive real part
A^T	Transpose of A
$A(s)^*$	$A(-s)^T$, conjugate transpose
$\hat{x}(j\omega)$	Fourier transform of the signal $x(t)$
$< x, y >$	$= \int_{-\infty}^{\infty} y(t)^T x(t) dt$
	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(j\omega)^* \hat{x}(j\omega) d\omega$
$\|x\|$	$= \sqrt{< x, x >}$

has a causal inverse. The interconnection is stable *if, additionally, the inverse is bounded.*

Theorem 1 (The IQC Theorem) *[3, 6] Let $G(s) \in \mathcal{RH}_\infty^{l \times m}$, and let $\Delta : \mathcal{L}_{2e}^l[0, \infty) \mapsto \mathcal{L}_{2e}^m[0, \infty)$ be a bounded causal operator. Assume that:*

i) for every $\alpha \in [0, 1]$, the interconnection of G and Δ_α is well-posed where Δ_α is a parameterization of Δ which satisfies

a) $\Delta = \Delta_\alpha|_{\alpha=1}$,

b) Δ_α is bounded and causal for $\alpha \in [0, 1]$,

c) there exists $\gamma > 0$ such that

$$\|\Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y)\| \leq \gamma |\alpha_1 - \alpha_2| \cdot \|y\| \quad (2)$$

for all $\alpha_1, \alpha_2 \in [0, 1]$,

ii) the interconnection of G and $\Delta_\alpha|_{\alpha=0}$ is stable,

iii) for every $\alpha \in [0, 1]$, the IQC defined by Π is satisfied by Δ_α, that is,

$$\left\langle \Pi \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix}, \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix} \right\rangle \geq 0, \quad (3)$$

iv) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \forall \omega \in \mathbb{R}. \quad (4)$$

Then, the feedback interconnection of G and Δ is stable.

It is often possible to use the linear parameterization $\Delta_\alpha = \alpha\Delta$. Then, conditions a), b) and c) of the above theorem can be omitted [3, 6].

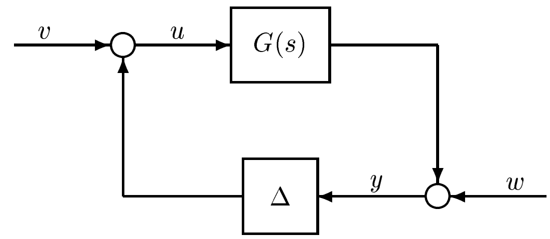

Figure 1: Basic feedback configuration

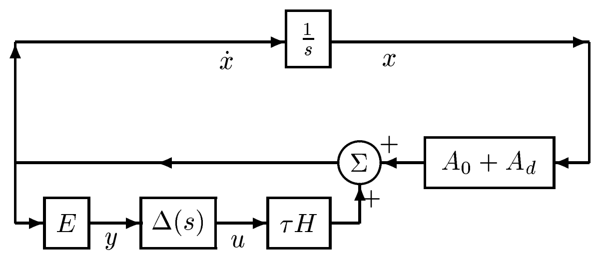

Figure 2: Time-delay system configuration in inverse uncertainty formulation where $\Delta(s) = \frac{1}{s\tau}(e^{-s\tau} - 1)$.

Lemma 1 (Kalman-Yakubovich-Popov Lemma) *[10] Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times k}$ and symmetric matrix $\Omega \in \mathbb{R}^{(n+k)\times(n+k)}$, there exists a symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that*

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \Omega < 0, \quad (5)$$

if and only if there exists some constant $\epsilon > 0$ such that

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \Omega \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} + \epsilon I \leq 0 \quad (6)$$

for all $\omega \in \mathbb{R}$.

4 Main Result

Lemma 2 *The system (1) is asymptotically stable if and only if $A_0 + A_d$ is Hurwitz and*

$$\mathcal{A}(s, \tau) = sI - (I - \tau\Delta(s)A_d)^{-1}(A_0 + A_d) \quad (7)$$

is nonsingular for all $s \in \mathbb{C}_+$, where

$$\Delta(s) = \frac{1}{s\tau}(e^{-s\tau} - 1) \quad (8)$$

Proof: The system (1) is asymptotically stable if and only if

$$\hat{A}(s, \tau) = sI - A_0 - A_d\, e^{-s\tau} \quad (9)$$

is nonsingular for all $s \in \mathbb{C}_+$. Suppose that $\hat{\mathcal{A}}(s,\tau)$ is singular for some $s \in \mathbb{C}_+$. Then, there exists a non-zero vector x such that

$$\begin{aligned} 0 &= \hat{\mathcal{A}}(s,\tau)x = (sI - A_0 - A_d\, e^{-s\tau})x \\ &= (sI - A_0 - A_d - A_d(e^{-s\tau} - 1))x \\ &= (sI - A_0 - A_d - \tau\Delta(s)A_d\, s)x \\ &= ((I - \tau\Delta A_d)s - (A_0 + A_d))\, x \\ &= \left(sI - (I - \tau\Delta(s)A_d)^{-1}(A_0 + A_d)\right) x \\ &= \mathcal{A}(s,\tau)x \end{aligned}$$

Whence, $\mathcal{A}(s,\tau)$ is singular some $s \in \mathbb{C}_+$, which proves sufficiency. Necessity can be proved in the same way. ∎

Let us decompose the matrix A_d as

$$A_d = HE, \quad H \in \mathbb{R}^{n\times q}, \;\; E \in \mathbb{R}^{q\times n} \qquad (10)$$

where $q \le n$, and H and E are of full rank.

With inverse uncertainty formulation and Lemma 2, we propose a sufficient condition for stability of the system (1), which is main theorem of this paper.

Theorem 2 (Main Theorem) *Suppose that* $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{q\times q}$ *and* $S \in \mathbb{R}^{q\times q}$. *Then, the system (1) is stable if there exist symmetric matrices* $P > 0$, $Q > 0$, *and a skew-symmetric matrix* S *such that*

$$\begin{bmatrix} A^TP + PA + C^TQC & PB + C^TS + C^TQD \\ B^TP + S^TC + D^TQC & -Q + D^TQD + S^TD + D^TS \end{bmatrix} < 0 \quad (11)$$

where

$$A = A_0 + A_d, \;\; B = \tau H, \;\; C = EA, \;\; D = \tau EH. \quad (12)$$

Proof: If we let the time-delay uncertainty be $\Delta(j\omega) = \frac{1}{j\omega\tau}(e^{-j\omega\tau} - 1)$, a state-space representation of the system $G(s)$ with u as a input and y as a output (see Fig. 1 and Fig. 2) can be expressed as

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx + Du, \\ u = \Delta(y) \end{cases} \qquad (13)$$

where

$$A = A_0 + A_d, \;\; B = \tau H, \;\; C = EA, \;\; D = \tau EH$$

By Lemma 2, the asymptotic stability of the time-delay system (1) is equivalent to that of the system (13).

If we set $\Pi = \begin{bmatrix} Q & S \\ S^T & -Q \end{bmatrix}$ and $\Delta_\alpha = \alpha\Delta$, we have

$$\begin{aligned} &\begin{bmatrix} I \\ \Delta_\alpha(j\omega) \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \Delta_\alpha(j\omega) \end{bmatrix} \\ &= Q - \alpha^2\Delta(j\omega)^*Q\Delta(j\omega) + \alpha S\Delta(j\omega) + \alpha\Delta(j\omega)^*S^T \\ &= Q - \alpha^2\Delta(j\omega)^*Q\Delta(j\omega) \\ &= Q(1 - \alpha^2\|\Delta(j\omega)\|^2) \\ &\ge 0 \end{aligned}$$

for all $\omega \in \mathbb{R}$ and $\alpha \in [0,1]$ since $S\Delta(j\omega) = 0$ and $\|\Delta(j\omega)\|_\infty \le 1$. $S\Delta(j\omega) = 0$ comes from the fact that $\Delta(j\omega)I$ is diagonal and S is skew-symmetric.

And we also have

$$\begin{aligned} &\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \\ &= \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}^* \Pi \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \\ &< 0 \end{aligned}$$

from the Lemma 1 and Eq. (11).

Thus, by Theorem 1, the system (1) is stable. ∎

We can find a upper bound on the delay margin τ_{max} for a system (1) by maximizing τ subject to (11) in Theorem 2. This is an LMI problem, which can be easily solved using software package.

5 Numerical Example

Consider the autonomous system of (1) with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & 0.25 \end{bmatrix}, \;\; A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.85 \end{bmatrix} \qquad (14)$$

which is the same example considered in Fu *et al.*[2]. The estimate of maximum delay margin using Theorem 2 is $\tau_{max} = 0.9999$. We used the `LMITOOL`[8] designed by ENSTA Optimization and Control Group in solving the LMI problem (11).

As comparisons, we can see that the estimate of allowable maximum time delay is $\tau_{max} = 0.6417$ when the Theorem 6 in [2] is used and it is $\tau_{max} = 0.9848$ when the Theorem 7 in [2] is used while the optimal value for the system with the given parameters, A_0 and A_d, is $\tau_{opt} = 1.54$[7]. We can see that our result is less conservative than [2] even without finding a SISO filter $f(s)$ such that $|f(j\omega)| \ge |\frac{sin(\omega)}{\omega}|$, $\forall\, \omega \in \mathbb{R}$ in order to apply the Theorem 7 in [2].

6 Discussion

We have derived a sufficient condition for asymptotic stability of time-delay system (1) in the form of an

LMI (11) while Fu *et al.*[2] provided two different LMI stability criteria for time-delay systems. The dimension of our LMI, $(n+q)\times(n+q)$, is less than those in [2], $(n+2q)\times(n+2q)$. Therefore, our LMI is superior to those in [2] from the computation point of view. We also performed Monte Carlo simulations which generated random matrix A_0 and A_d which satisfy the Hurwitz condition of $A = A_0 + A_d$ and compared the results from our LMI and LMI in [2]. We found that 578 out of 600 cases produced less conservative results than the method of [2].

7 Conclusion

We have presented a delay-dependent stability criterion for a continuous-time system with single MIMO time-delay in states. This is a sufficient condition expressed in the form of LMI, which can be solved using standard computer software tools. Monte Carlo simulations demonstrated that in 96% of the cases considered our approach was less conservative than that of [2]

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