

Multipliers for Repeated MIMO Nonlinearities ¹

Vishwesh V. Kulkarni
 Electrical Engineering—Systems
 University of Southern California
 Los Angeles, CA 90089-2563
 vishwesh@usc.edu

Michael G. Safonov
 Electrical Engineering—Systems
 University of Southern California
 Los Angeles, CA 90089-2563
 msafonov@usc.edu

Abstract

The largest class of linear MIMO convolution operators that preserves positivity of repeated MIMO, incrementally positive, norm-bounded, memoryless nonlinearities is obtained.

1 Introduction

More often than not, stability analysis problems which feature repeated norm-bounded incrementally positive single input single output (SISO or scalar) or multi input multi output (MIMO) nonlinearities are encountered — e.g. channels having standard saturation nonlinearity or quantization nonlinearities among others. A key step in multiplier based stability analysis of such systems is to characterize a multiplier, say M , which *preserves positivity* of such a nonlinearity N in the sense that $N \geq 0$ implies $MN \geq 0$ (see [1, Theorem 1,2], [2], [3] and references therein for a relevant discussion).

The latest and, by far, the most comprehensive result on such repeated scalar nonlinearities is due to D’Amato et al [2]. In that work, sufficiency conditions, under which MIMO convolution operators preserve positivity of such nonlinearities, are derived using a creative manipulation of Lemma 7 of [1] (see Lemma 1 of [2]). However, their characterization is incomplete since necessity of these conditions has not been proved. In this note, we give a complete characterization of MIMO convolution operators preserve positivity of such nonlinearities and readily extend that result to the repeated MIMO case. To that end, the results developed in [4] to characterize bounded convolution operators which preserve positivity of norm bounded, monotonic scalar nonlinearities are used as the starting point and, essentially, the approach used in [4, Ch. 3] is generalized for our purposes.

This paper is organized as follows. In Section 2, necessary terminology is introduced and the problems are formally posed in Section 3. Background results are

noted down in Section 4. Main results are presented in Section 5 and discussed in Section 6. The technical note is concluded in Section 7.

2 Preliminaries

The notation used is summarized in Table 1. Capital letter symbols, e.g. F and G , denote operators whereas small letters, e.g. x and y , denote real signals which may possibly be vector-valued or matrix-valued. \mathcal{Z} denotes the set of all integers; (\cdot) denotes conjugate transpose of a vector or matrix (\cdot) . A sequence $\{x(k)\}_{k=-\infty}^{\infty}$ is noted down simply as $\{x\}$. The vector space ℓ_2^p is referred to as ℓ_2 unless an explicit distinction between the two spaces needs to be made. To save space, the statements of the form “A related to B” and “C related to D” are written as “A (C) related to B (D)”.

Table 1: Notation

Symbol	Meaning
\mathbb{R} (\mathbb{C}, \mathbb{R}^+)	Set of all real (complex, nonnegative real) numbers.
$\text{herm}(m)$	$= \frac{1}{2}(m + \bar{m}^T)$, for $m \in \mathbb{C}^{n \times n}$ or $\mathbb{R}^{n \times n}$.
$\langle x, y \rangle$	$= \sum_{k=-\infty}^{\infty} y(k)^T x(k)$.
$\ x\ $	$= \sqrt{\langle x, x \rangle}$.
$\ x\ _1$	$= \sum_{k=-\infty}^{\infty} x(k) $ (for scalar valued signals).
ℓ_2	Space of possibly vector valued signals x for which $\ x\ _2$ exists.
\hat{x}	$= \sum_{k=-\infty}^{\infty} x(k)z^{-k}$, (z-transform).

Definition 1 [Sequences: similarly ordered, unbiased] Two sequences of real numbers $\{x_1, x_2, \dots, x_n, \dots\}$ and $\{y_1, y_2, \dots, y_n, \dots\}$, are said to be *similarly ordered* if the inequality $x_i < x_j$ implies $y_i \leq y_j$ for all i, j . The sequences are said to be *unbiased* if $x_i y_i \geq 0 \forall i$. The sequences are said to be *similarly ordered and symmetric* if they are unbiased and, in addition, the sequences $\{|x_i|\}$ and $\{|y_i|\}$ are similarly ordered.

¹This research is supported by AFOSR F49620-98-1-0026.

Definition 2 [hyperdominance, dominance]

An operator $A : \ell_2 \rightarrow \ell_2$ is said to be *doubly hyperdominant* if the elements m_{ij} of its associated matrix M has the following properties.

1. $m_{ij} \leq 0, \forall i \neq j$; and
2. $0 \leq \sum_{i=1}^n m_{ij} < \infty, 0 \leq \sum_{j=1}^n m_{ij} < \infty \quad \forall i, j$.

It is said to be *doubly dominant* if in addition to the properties 1 and 2, it also holds that

$$m_{ii} \geq \sum_{j=1, j \neq i}^n |m_{ij}|, \quad m_{ii} \geq \sum_{j=1, j \neq i}^n |m_{ji}| \quad \forall i.$$

Remark 1 Definitions 1-2 are as given in [4]. ■

Definition 3 [Associated operator]

Given an operator $M : \ell_2^p \rightarrow \ell_2^p$ that maps a sequence $\{x\} \in \ell_2^p$ into a sequence $\{y\} \in \ell_2^p$, its *associated operator* $\tilde{M} : \ell_2 \rightarrow \ell_2$ maps the sequence $\{\tilde{x}\} \in \ell_2$ into the sequence $\{\tilde{y}\} \in \ell_2$ where $\{\tilde{x}\} \doteq \{\dots, x_1(k), x_2(k), \dots, x_p(k), x_1(k+1), \dots\}$ and $\{\tilde{y}\} \doteq \{\dots, y_1(k), y_2(k), \dots, y_p(k), y_1(k+1), \dots\}$, $k \in \mathcal{Z}$. The matrix associated with \tilde{M} is given by

$$M \doteq \begin{bmatrix} \bar{m}_{0,0} & \bar{m}_{0,1} & \bar{m}_{0,2} & \bar{m}_{0,3} & \dots & \dots \\ \bar{m}_{1,0} & \bar{m}_{1,1} & \bar{m}_{1,2} & \bar{m}_{1,3} & \bar{m}_{1,4} & \ddots \\ \bar{m}_{2,0} & \bar{m}_{2,1} & \bar{m}_{2,2} & \bar{m}_{2,3} & \bar{m}_{2,4} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

where $\bar{m}_{k,l} \in \mathbb{R}^{p \times p}$ such that (i, j) -th element of $\bar{m}_{k,l}$ is the coefficient of $x_j(l)$ in the expression $y = Mx$ for $y_i(k)$. If $\bar{m}_{k,l} = \bar{m}_{k+n, l+n} \quad \forall k, l, n \in \mathcal{Z}$ then M is called *block Toeplitz* type operator. M is said to be a *convolution operator* if the matrix entries $m_{k,l}$ are completely described by an array $\{m_{k-l}\}$ so that for all $\{x\}$ and $\{y\}$,

$$y(k) = \sum_{l=-\infty}^{\infty} m_{k,l} x(l) = \sum_{l=-\infty}^{\infty} m_{k-l} x(l) \quad \text{for all } k, l.$$

Definition 4 An operator $F : \ell_2 \rightarrow \ell_2$ is said to be *positive* if $\langle x, Fx \rangle \geq 0 \quad \forall x \in \ell_2$. It is said to be *bounded* if there exists $\gamma \in \mathbb{R}^+$ such that $\|Fx\| \leq \gamma \quad \forall x \in \ell_2$.

Definition 5 The class of positive operators $N : \ell_2 \rightarrow \ell_2$ which are *incrementally positive*, i.e.

$$\langle x - y, N(x) - N(y) \rangle \geq 0 \quad \forall x, y \in \ell_2, \quad (1)$$

and have a *finite gain*, i.e.

$$\exists C \in \mathbb{R}^+ \ni \|N(x) - N(y)\| \leq C \|x - y\| \quad \forall x, y \in \ell_2, \quad (2)$$

is denoted \mathcal{N}_{inc} . $\mathcal{N} \doteq \{N \in \mathcal{N}_{inc} | N(0) = 0\}$. $\mathcal{N}_{odd} \doteq \{N \in \mathcal{N} | N(x) = -N(-x) \quad \forall x\}$.

Definition 6 \mathcal{N}^{RS} denotes the class of MIMO nonlinearities $\underline{N} : \ell_2^n \rightarrow \ell_2^n$ defined as

$$\underline{N}(\zeta) \doteq [N(\zeta_1) \ N(\zeta_2) \ \dots \ N(\zeta_n)]^T \quad (3)$$

where $N \in \mathcal{N}, N : \ell_2 \rightarrow \ell_2$. A shorthand notation for (3) is $\underline{N} = \text{diag}(N)$. Replacing \mathcal{N} in the definition of \mathcal{N}^{RS} by \mathcal{N}_{odd} , \mathcal{N}_{odd}^{RS} is obtained.

Definition 7 \mathcal{N}^{RM} denotes the class of MIMO nonlinearities $\underline{N} : \ell_2^n \rightarrow \ell_2^n$ defined as

$$\underline{N}(\zeta) \doteq [N(\zeta^{[1]}) \ N(\zeta^{[2]}) \ \dots \ N(\zeta^{[m]})]^T \quad (4)$$

where $N \in \mathcal{N}, N : \ell_2^l \rightarrow \ell_2^l$, and $\zeta \in \mathbb{R}^n$ is decomposed as

$$\zeta = [\zeta^{[1]} \ \zeta^{[2]} \ \dots \ \zeta^{[m]}]$$

where $\zeta^{[l]} \doteq [\zeta_{(i-1)l+1} \ \zeta_{(i-1)l+2} \ \dots \ \zeta_{il}]$, $l \doteq n/m$. A shorthand notation for (4) is $\underline{N} = \text{diag}(N)$. Replacing \mathcal{N} in the definition of \mathcal{N}^{RM} by \mathcal{N}_{odd} , \mathcal{N}_{odd}^{RM} is obtained.

Remark 2 \mathcal{N}^{RS} is the class of repeated scalar nonlinearities whereas \mathcal{N}^{RM} is the class of repeated MIMO nonlinearities. ■

Definition 8 \mathcal{M}_{odd}^{RS} denotes the class of MIMO convolution operators $M : x \mapsto m * x$ with z -transform

$$\hat{m}(z) \doteq g - \hat{h}(z) \quad (5)$$

where $g \in \mathbb{R}^{n \times n}$, $\hat{h} : \mathbb{R} \mapsto \mathbb{R}^{n \times n}$ such that

$$g_{ii} \geq \sum_{i=1}^n |g_{ij}| + \sum_{i=1}^n \|h_{ij}\|_1 \quad \forall i = 1, 2, \dots, n. \quad (6)$$

The subclass \mathcal{M}^{RS} is obtained by further stipulating

$$g_{ij} \leq 0 \quad \forall i \neq j, \quad h_{ij}(k) \geq 0 \quad \forall i, j, k. \quad (7)$$

Under the further restriction

$$g, h \text{ are symmetric matrices,} \quad (8)$$

\mathcal{M}^D (\mathcal{M}_{odd}^D) is obtained from \mathcal{M}^{RS} (\mathcal{M}_{odd}^{RS}).

Remark 3 \mathcal{M}^D and \mathcal{M}_{odd}^D are useful classes of multipliers derived by D'Amato et al (see [2]) for the analysis of repeated scalar nonlinearities. ■

Remark 4 The eigenvalues of the frequency response $\widehat{m}(z)$ of an M satisfying (5)-(7) lie outside the unit circle (see [5, Theorem 6.1.1, pp. 344]) for all $|z| = 1$. Therefore, $\text{herm}(\widehat{m}(z)) \geq 0 \forall z$ for every $M \in \mathcal{M}^{RS}$. If M is further restricted to be in \mathcal{M}^D then $\widehat{m}(z)$ is a Hermitian matrix for all $|z| = 1$. ■

Definition 9 \mathcal{M}^{RM} (\mathcal{M}_{odd}^{RM}) denotes the class of MIMO convolution operators obtained by taking the Kronecker product of elements of \mathcal{M}^{RS} (\mathcal{M}_{odd}^{RS}) with an identity matrix of suitable size i.e.

$$\mathcal{M}^{RM} \doteq \{M \otimes I \mid M \in \mathcal{M}^{RS}\}.$$

3 Problem Formulation

The problems of interest are as follows.

Problem 1 Find the largest class of bounded convolution operators that preserves positivity of *every* non-linearity in \mathcal{N}^{RS} (\mathcal{N}_{odd}^{RS}).

Problem 2 Find the largest class of bounded convolution operators that preserves positivity of *every* non-linearity in \mathcal{N}^{RM} (\mathcal{N}_{odd}^{RM}).

4 Background Results

Sufficiency conditions for positivity preservation of \mathcal{N}^{RS} have been recently derived by D'Amato et al. Paraphrasing for ease of notation, their main result, viz. Theorem 1 (see [2]), is as follows.

Lemma 1 [2, D'Amato et al]
 \mathcal{M}^D (\mathcal{M}_{odd}^D) is positivity preserving for \mathcal{N}^{RS} (\mathcal{N}_{odd}^{RS}).

The above result is derived as a sufficiency condition and not as a necessary condition. Therefore, the multiplier characterization given is partial. In this regard, it is worthwhile to note the following interesting result, established in the scalar case, by Willems (see [4, Theorem 3.11, pp. 63]). For easy reading, its statement is slightly modified.

Lemma 2 [4, Willems]

Let $M : \ell_2 \rightarrow \ell_2$ be a bounded operator. Then, $\langle x, My \rangle$ is non negative for all similarly ordered unbiased (similarly ordered symmetric unbiased) sequences $\{x\}, \{y\} \in \ell_2$ if and only if M is doubly hyperdominant (doubly dominant). ■

5 Main Result

Based on Lemma 2, the complete characterization of linear positivity preserving multipliers for \mathcal{N}^{RS} and for \mathcal{N}_{odd}^{RS} is given as follows.

Theorem 1 A bounded operator $M : \ell_2^p \rightarrow \ell_2^p$ preserves positivity of every $N \in \mathcal{N}^{RS}$ ($N \in \mathcal{N}_{odd}^{RS}$) if and only if the matrix associated with its associated operator \tilde{M} is doubly hyperdominant (doubly dominant).

Proof: We shall prove the result for \mathcal{N}^{RS} . The case for \mathcal{N}_{odd}^{RS} follows on similar lines.

An $N \in \mathcal{N}^{RS}$ can be expressed as $N = \text{diag}(\phi)$ where $\phi \in \mathcal{N}$, ϕ scalar. Given sequences $\{x_i\}$ of real numbers, define $y_i \doteq \phi(x_i)$, $i = 1, 2, \dots, p$. Let $\tilde{x}(k) \doteq [x_1(k) \ x_2(k) \ \dots \ x_p(k)]^T$, $\tilde{y}(k) \doteq [y_1(k) \ y_2(k) \ \dots \ y_p(k)]^T$ for all $k \in Z$. Note that the sequences $\{x_i\}$ and $\{y_i\}$ are similarly ordered and unbiased for all i since $N \in \mathcal{N}$. Therefore, the sequences $\{\tilde{x}\}$ and $\{\tilde{y}\}$ are similarly ordered and unbiased. Observe that

$$\langle x, AN(x) \rangle = \langle \tilde{x}, \tilde{A}\tilde{y} \rangle.$$

Since \tilde{A} is bounded, \tilde{A} is a bounded operator as well. The result then follows using Lemma 2. QED.

We now state the main results.

Theorem 2 [Solution to Problem 1]

A bounded convolution operator $A : \ell_2^p \rightarrow \ell_2^p$ preserves positivity of every $N \in \mathcal{N}^{RS}$ ($N \in \mathcal{N}_{odd}^{RS}$) if and only if $M \in \mathcal{M}^{RS}$ ($M \in \mathcal{M}_{odd}^{RS}$).

Proof: We shall prove the result for \mathcal{N}^{RS} . The case for \mathcal{N}_{odd}^{RS} follows on similar lines.

If A is a convolution operator, then its associated operator \tilde{A} is block Toeplitz type with the elements $\tilde{m}_{k,l}$ of its associated matrix \tilde{M} completely determined by the array $\{\tilde{m}_{k-l}\}$ as $\tilde{m}_{k,l} = \tilde{m}_{k-l} \forall k, l$. By Theorem 1, A is positivity preserving if and only if \tilde{M} is doubly hyperdominant. Since \tilde{M} is block Toeplitz, the double hyperdominance conditions need only be checked on a block of p columns and on a block of p rows. The conditions are precisely the ones given by (6) and (7). QED.

Remark 5 Restricting attention to *symmetric* convolution operators, it follows that \mathcal{M}^D (\mathcal{M}_{odd}^D) is the largest class of convolution operators that preserves positivity of \mathcal{N}^{RS} (\mathcal{N}_{odd}^{RS}); so that *under the restriction* that the convolution operator be symmetric, the characterization given by D'Amato et al is actually a complete characterization. ■

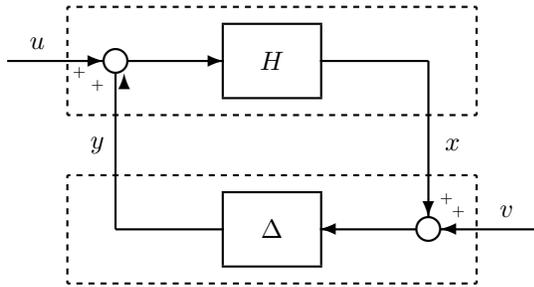


Figure 1: The feedback system \mathcal{S} . H and Δ are operators; the disturbances u, v enter the system additively. H is stable and linear time invariant; Δ is a (block) diagonal operator having nonlinearities, time-varying elements and parametric uncertainties on its diagonal.

Theorem 2 can be extended to cover the case of repeated MIMO nonlinearities as follows.

Theorem 3 [Solution to Problem 2]

A bounded convolution operator $A : \ell_2^p \rightarrow \ell_2^p$ preserves positivity of every $N \in \mathcal{N}^{RM}$ ($N \in \mathcal{N}_{odd}^{RM}$) if and only if $M \in \mathcal{M}^{RM}$ ($M \in \mathcal{M}_{odd}^{RM}$).

Proof: Follows using arguments used in the proof of Theorem 1. QED.

6 Discussion

A feedback system can be viewed as a canonical system \mathcal{S} (see Fig. 1) in which the feedforward element H is a bounded, causal, LTI transfer function in RL_∞ and the feedback element Δ is an uncertain element such as a nonlinearity or a time-varying element or a parametric uncertainty; in general, Δ can be a (block) diagonal collection of such elements. Stability of \mathcal{S} is equivalent to the existence of a quadratic functional which separates the graph of Δ from the inverse graph of H . A useful technique to test this computationally involves characterizing integral quadratic constraints (IQCs) satisfied by Δ (see [6] for a detailed discussion on IQCs). To reduce conservatism in stability analysis, the IQC characterization should be a snug fit, i.e. it should be based on ‘if and only if’ conditions.

The characterization given by D’Amato et al is a sufficiency condition only. Therefore, it is conceivable that there are realizations of \mathcal{S} , having $\Delta \in \mathcal{N}^{RS}$, that are stable, and yet their stability can not be concluded using multipliers in \mathcal{M}^D .

The results presented in this manuscript are for dis-

crete time systems; but the results can be stated for continuous time systems as well, after a slight technical modification.

7 Conclusion

The largest class of MIMO convolution operators that preserves positivity of repeated MIMO, monotone, norm-bounded, memoryless nonlinearities is obtained. It is shown that, the class of multipliers characterized by D’Amato et al [2] is the largest class of *symmetric* MIMO convolution operators that preserves positivity of repeated scalar, monotone, norm-bounded, memoryless nonlinearities.

References

- [1] G. Zames and P. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM J. Control and Optimization*, 6:89–108, 1968.
- [2] F. D’Amato, A. Megretski, U. Jönsson, and M. Rotea. Integral quadratic constraints for monotonic and slope-restricted diagonal operators. In *Proc. American Control Conf.*, pages 2375–2379, San Diego, CA, June 2–4, 1999. IEEE Press, New York.
- [3] M.G. Safonov and V.V. Kulkarni. Zames-Falb multipliers for MIMO nonlinearities. In *Proc. American Control Conf.*, pages 4144–4148, Chicago, IL, June 28–30, 2000. IEEE Press, New York. Also, to appear in the Nov 2000 issue of the International Journal on Robust and Nonlinear Control. URL — <ftp://roth.usc.edu/pub/safonov/safo00a.pdf>.
- [4] J. Willems. *The Analysis of Feedback Systems*. The MIT Press, Cambridge, MA, 1973.
- [5] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, New York, NY, 1985.
- [6] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Autom. Control*, AC-42(6):819–830, June 1997.