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Incremental Positivity Preservation Properties of Zames-Falb Multipliers [1]

Vishwesh V. Kulkarni
Electrical Engineering—Systems
University of Southern California
Los Angeles, CA 90089-2563
vishwesh@usc.edu

Michael G. Safonov
Electrical Engineering—Systems
University of Southern California
Los Angeles, CA 90089-2563
msafonov@usc.edu

Abstract

Zames-Falb multipliers, known to preserve positivity of incrementally positive nonlinearities, may fail to preserve incremental positivity thereof.

1 Introduction

A class $\mathcal{M}$ of multipliers was introduced by Zames and Falb in [1] to reduce conservatism in stability analysis of a feedback system $\mathcal{S}$ having an $H \in \mathcal{H}$ in the feedforward path and a nonlinearity $N \in \mathcal{N}$ in the feedback path where $\mathcal{H}$ is the class of stable, linear time-invariant (LTI) transfer functions and $\mathcal{N}$ is the class of memoryless, monotone, norm-bounded nonlinearities. A key property of $\mathcal{M}$ (see [1] for more properties) is that it is *positivity preserving* for $\mathcal{N}$ in the sense that $M \in \mathcal{M}$ implies positivity of the operator M^*N for any $N \in \mathcal{N}$. The resulting stability characterization is the sharpest possible in SISO setting and, furthermore, stability conditions prescribed by it can be tested using software tools (see [2] and [3]).

Recently, Zames-Falb stability analysis approach has been extended to the multivariable case, viz. for the systems with repeated scalar nonlinearities (see [4]) and for the systems with full-block MIMO nonlinearities (see [5]). However, whether $\mathcal{M}$ is *incremental positivity preserving* for $\mathcal{N}$ has been a topic left unaddressed. The issue is of significance since the lack of incremental positivity preservation property implies that the Zames-Falb multipliers may not be useful in deducing *incremental stability* of a given system. In other words, the lack of incremental positivity preservation property implies that even if a Zames-Falb multiplier estabilishes stability of the given system, the system may be a discontinuous relation between two L_2-spaces, i.e. it may yield an unbounded incremental output for a bounded incremental input. In this paper, this point is investigated and we show that $\mathcal{M}$ *can fail* to preserve incremental positivity of a nonlinearity in $\mathcal{N}$.

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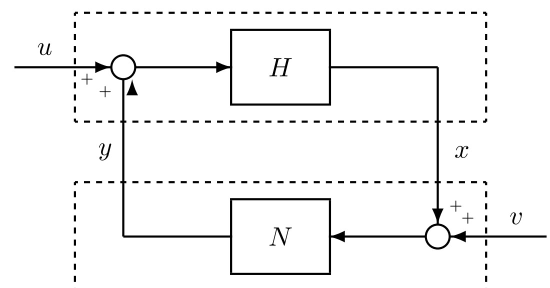

Figure 1: The feedback system $\mathcal{S}$. H is stable and linear time invariant; N is memoryless, norm bounded and incrementally positive.

This paper is organized as follows. In Section 2, necessary terminology is introduced. The main results are presented in Section 3 and briefly discussed in Section 4. The technical note is concluded in Section 5.

2 Preliminaries

Notation used is summarized in Table 1. Capital letter symbols, e.g. F and G, denote operators whereas small letters, e.g. x and y, denote real signals which may possibly be vector-valued or matrix-valued. $\mathcal{Z}$ denotes the set of all integers and $\bigcup$ is the set union operator; $\overline{(\cdot)}$ denotes conjugate transpose of a vector or matrix $(\cdot)$ and Re$(\cdot)$ (Im$(\cdot)$) denotes real (imaginary) part of $(\cdot)$; trace$[m]$ denotes trace of a square matrix m. For scalar valued x, $||x||_1 \doteq \int_{-\infty}^{\infty} |x(t)|dt$. The vector space $L_2^n\ (\mathbb{R}^n)$ is referred to as $L_2\ (\mathbb{R})$ unless an explicit distinction between the two spaces needs to be made.

Definition 1 An operator $F : L_2 \longrightarrow L_2$ is said to be *positive* if $\langle x, Fx\rangle \geq 0\ \forall x \in L_2$. It is said to be *incrementally positive* if

$$\langle x - y, N(x) - N(y)\rangle \ \geq 0 \ \ \forall x, y \in L_2. \tag{1}$$

Table 1: Notation

Symbol	Meaning
$\mathbb{R}$ ($\mathbb{C}, \mathbb{R}^+$)	Set of all real (complex, nonnegative real) numbers.
skew(m)	$= \frac{1}{2}(m - \overline{m}^T)$, for $m \in \mathbb{C}^{n\times n}$ or $\mathbb{R}^{n\times n}$.
$\langle x, y\rangle$	$= \int_{-\infty}^{\infty} y(t)^* x(t) dt.$
$\|\|x\|\|$	$= \sqrt{\langle x, x\rangle}.$
L_2	Space of possibly vector valued signals x for which $\|\|x\|\|_2$ exists.
$x(t) * y(t)$	$= \int_{-\infty}^{\infty} x(\tau) y^T(t-\tau) d\tau$, (convolution).
x^*	$x^*(t) = x^T(-t)$ if x is a real signal.
$r_{xy}(t)$	$= x * y^* = \int_{-\infty}^{\infty} x(t+\tau) y^T(\tau) d\tau$, (correlation function).
$\widehat{x}$	$= \mathcal{F}x = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$, (Fourier transform).
$\mathcal{C}^1$	Set of continuously differentiable maps.
$\delta(t)$	$= \begin{cases} 1 & t = 0, \\ 0 & \text{else.} \end{cases}$ (delta function).
D_τ	$(D_\tau x)(t) \doteq x(t - \tau)\ \forall t$ ($\tau \in \mathbb{R}$) (time-shift operator).

Definition 2 An operator $G : L_2 \to L_2$ is said to be *KS-positive* if given any $x, y \in L_2$,

$$\langle x - y, G(x) - G(y)\rangle \ \geq 0 \tag{2}$$

under the restriction that either $x \equiv \zeta$ or $y \equiv \zeta$ for some $\zeta \in \mathbb{R}$.

Definition 3 The class of incrementally positive nonlinearities, N, which have finite incremental gain i.e.

$$\exists C \in \mathbb{R}^+ \ni \|N(x) - N(y)\| \leq C\|x - y\| \ \ \forall x, y \in L_2 \tag{3}$$

is denoted $\mathcal{N}_{inc}$ and $\mathcal{N} \doteq \{N \in \mathcal{N}_{inc} | N(0) = 0\}$.

Definition 4 $\mathcal{M}_{odd}$ denotes the class of transfer functions (convolution operators) $M : x \mapsto m * x$ with

$$\widehat{m}(j\omega) \doteq m_0 - \widehat{z}(j\omega) \ \ \forall\omega \tag{4}$$

$$\text{where} \ \ m_0 - ||z||_1 > 0. \tag{5}$$

Subclass $\mathcal{M}$ obtained under the restriction $z(t) \geq 0\ \forall t$ is the class of *Zames-Falb multipliers*. $\mathcal{M}_{odd}$ is useful in analysis for odd nonlinearities in $\mathcal{N}$ (see [1] and [5]).

Definition 5 A relation $\Xi : L_2 \to L_2$ is said to be *incrementally stable* if it is

1. *continuous* i.e. given $x \in L_2$ and $\Delta > 0$, $\exists\delta > 0 \ni ||x - y|| < \delta \Longrightarrow ||\Xi x - \Xi y|| < \Delta\ \forall y \in L_2$; and,
2. *bounded* i.e. given $x \in L_2\ \exists M > 0 \ni ||\Xi x|| < M$.

Remark 1 Definition 5 is as given in [6]. ∎

3 Main Result

We shall first establish a key preliminary result.

Lemma 1 Suppose $N \in \mathcal{N}_{inc} \cap \mathcal{C}^1, \tau \in \mathbb{R}, \tau \neq 0$. Let $\tilde{x} \doteq x - \overline{x}$ and $\tilde{y} \doteq N(x) - N(\overline{x})$. Then,

$$\text{trace}[r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau)] \geq 0 \qquad \forall x, \overline{x} \in L_2 \tag{6}$$

if and only if N is linear. Furthermore,

$$\text{trace}[2r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau) - r_{\tilde{y}\tilde{x}}(-\tau)] \geq 0 \ \ \forall x, \overline{x} \in L_2 \tag{7}$$

if and only if N is linear.

Proof:
(**if**) N linear $\Longrightarrow \tilde{y} = A\tilde{x}$ for a matrix A of suitable size. Hence,

$$\text{trace}[r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau)] = \langle \tilde{x}, (I - D_\tau)^* A\tilde{x}\rangle\,,$$
$$\text{trace}[2r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau) - r_{\tilde{y}\tilde{x}}(-\tau)] = \langle \tilde{x}, (2I - D_\tau - D_{-\tau})^* A\tilde{x}\rangle\,.$$

Since $(I - D_\tau)^* A$ and $(2I - D_\tau - D_{-\tau})^* A$ are nonnegative operators (see [5, Lemma 1]). Hence, (6) and (7) hold.

(**only if**) Suppose $\tau > 0$. It suffices to produce a scalar case instance of $\{N \in \mathcal{N}_{inc} \cap \mathcal{C}^1, x \in L_2, \overline{x} \in L_2, \tau \in \mathbb{R}\}$ as a counter-example for which (6) does not hold. In that case, it may easily be verified that (7) does not hold as well.

First, observe that if a scalar $N \in \mathcal{N}_{inc} \cap \mathcal{C}^1$ is not linear, then there exist two points $\zeta_1, \zeta_2 \in \mathbb{R}$, $\zeta_1 \neq \zeta_2$ such that

$$\beta \doteq N'(x)\Big|_{x\,=\,\zeta_1} \neq \alpha \doteq N'(x)\Big|_{x\,=\,\zeta_2}$$

where $\alpha, \beta \geq 0$ since $N \in \tilde{\mathcal{N}}$. We need to demonstrate the existence of signals $x, \overline{x}$ for which (6) fails to hold. For that purpose, choose the test signals $x, \overline{x}$ to be pulse trains defined as follows.

$$\overline{x}(t) = \begin{cases} \zeta_1 & \text{if } t \in \mathcal{T}_1 \\ \zeta_2 & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} ; \ \ x(t) = \begin{cases} \overline{x}(t) + \epsilon\mu & \text{if } t \in \mathcal{T}_1 \\ \overline{x}(t) + \epsilon & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} ;$$

where, with $k \in \mathcal{Z}$, $\mathcal{T}_1 \doteq \bigcup_{k=0}^{\ell-1} [2k\tau, (2k+1)\tau)$, and $\mathcal{T}_2 \doteq \bigcup_{k=0}^{\ell-1} [(2k+1)\tau, (2k+2)\tau)$, $\mu \in \mathbb{R}$; the choices of $\epsilon > 0$ which is arbitrarily small, and $\ell > 0$ will be explained shortly.

With the above choice of x and $\overline{x}$, we have

$$\tilde{x}(t) = \begin{cases} \epsilon\mu & \text{if } t \in \mathcal{T}_1 \\ \epsilon & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} ; \ \ \tilde{y}(t) = \begin{cases} \beta\epsilon\mu + O(\epsilon^2) & \text{if } t \in \mathcal{T}_1 \\ \alpha\epsilon + O(\epsilon^2) & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} .$$

Let

$$Q \doteq \begin{bmatrix} \beta & 0 & 0 & 0 & \dots & 0 & 0 \\ -\beta & \alpha & 0 & 0 & \dots & 0 & 0 \\ 0 & -\alpha & \beta & 0 & \dots & 0 & 0 \\ 0 & 0 & -\beta & \alpha & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \beta & 0 \\ 0 & 0 & 0 & 0 & \dots & -\beta & \alpha \end{bmatrix}, \; Q \in \mathbb{R}^{2\ell \times 2\ell}$$

and $w \doteq [\mu \; 1 \; \mu \; 1 \; \dots \; \mu \; 1]^T, \; w \in \mathbb{R}^{2\ell}$. Now,

$$\begin{aligned} \text{trace}[r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau)] &= \langle \tilde{y}, (I - D_\tau)\tilde{x} \rangle \\ &= \epsilon^2 \tau \; w^T Q w + O(\epsilon^3) \\ &= \epsilon^2 \tau \; f_1(\mu) \; + \; \epsilon^2 (\ell - 1)\tau \; f_2(\mu) \; + \; O(\epsilon^3), \end{aligned}$$

where $f_1(\mu) \doteq \beta\mu^2 - \beta\mu + \alpha$, and

$$f_2(\mu) \;\; \doteq \;\; \beta\mu^2 - (\alpha + \beta)\mu + \alpha$$

Therefore,

$$\begin{aligned} &\text{trace}[r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau)] \\ &\quad = \;\; (\ell - 1)\tau\epsilon^2 \; [O(1/\ell) + f_2(\mu) + O(\epsilon)]. \end{aligned} \tag{8}$$

Now observe that $f_2(\cdot)$ is minimized w.r.t. μ when $\mu = \frac{\alpha+\beta}{2\beta}$, at which point its value $-\frac{(\alpha-\beta)^2}{4\beta}$ is strictly negative since $\alpha, \beta \geq 0, \alpha \neq \beta$. Then, choosing ℓ sufficiently large and ϵ sufficiently small, (8) implies $\text{trace}[r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(\tau)] < 0$. The case for $\tau < 0$ may be proved on similar lines. QED.

Theorem 1 Given an $N \in \mathcal{N}_{inc} \cap \mathcal{C}^1$ that is *not linear*, there exists an $M \in \mathcal{M}$ which does not preserve its incremental positivity.

Proof: It suffices to prove the result for a scalar N. Write $y \doteq N(x), \overline{y} \doteq N(\overline{x}), \tilde{x} \doteq x - \overline{x}$ and the corresponding $\tilde{y} \doteq y - \overline{y}$ where $x, \overline{x} \in L_2$. Given an $M \in \mathcal{M}$,

$$\begin{aligned} \langle M^* \left(N(x) - N(\overline{x}) \right), x - \overline{x} \rangle &= \langle m_0^* \tilde{y}, \tilde{x} \rangle - \langle z^* * \tilde{y}, \tilde{x} \rangle \\ &= \;\; \text{trace}[m_0 r_{\tilde{y}\tilde{x}}(0)] - \text{trace}[\int_{-\infty}^{\infty} z(-t) r_{\tilde{y}\tilde{x}}(-t) dt] \\ &= \;\; \left(m_0 - \int_{-\infty}^{\infty} z(t) dt \right) \text{trace}[r_{\tilde{y}\tilde{x}}(0)] \\ &\quad + \int_{-\infty}^{\infty} z(t) \left(\text{trace}[r_{\tilde{y}\tilde{x}}(0) - r_{\tilde{y}\tilde{x}}(t)] \right) dt. \end{aligned} \tag{9}$$

Conditions (1) and (3) ensure that $\gamma_1 \doteq \text{trace}[r_{\tilde{y}_* \tilde{x}_*}(0)]$ is a bounded positive quantity and since $M \in \mathcal{M}$, $m_0 - \int_{-\infty}^{\infty} z(t) dt \geq 0$. Now we use Lemma 1 to observe that there exists $\{N_* \in \mathcal{N}_{inc}, x_* \in L_2, \overline{x}_* \in L_2, t_* \in \mathbb{R}\}$ such that

$$\gamma_2 \doteq \text{trace}[r_{\tilde{y}_* \tilde{x}_*}(0) - r_{\tilde{y}_* \tilde{x}_*}(-t_*)] < 0 \tag{10}$$

where $\tilde{x}_* \doteq x_* - \overline{x}_*$ with the corresponding $\tilde{y}_* \doteq N(x_*) - N(\overline{x}_*)$. Choose the parameters $\{m_0, z(t)\}$ of the candidate multiplier M_* to be

$$\begin{aligned} &m_0 = 1, \; z(t) = \gamma_3 \; \delta(t - t_*) \\ &\text{where} \;\; 0 \leq \frac{\gamma_1}{\gamma_1 - \gamma_2} < \gamma_3 < 1. \end{aligned} \tag{11}$$

Note that $z(t) \geq 0 \; \forall t$ and $m_0 - ||z||_1 > 0$ so that $M_* \in \mathcal{M}$. Using (9)–(11), it may be verified that

$$\langle M_*^* \left(N_*(x_*) - N_*(\overline{x}_*) \right), x_* - \overline{x}_* \rangle < 0.$$

Thus, an element of the set $\{M \in \mathcal{M}, x \in L_2, \overline{x} \in L_2, N \in \mathcal{N}_{inc}\}$ such that $M^* N$ is not incrementally positive is proved to exist. QED.

Corollary 1 $\mathcal{M}_{odd}$ is incremental positivity preserving for every linear $N \in \mathcal{N}_{inc}$.

Proof: Follows on the lines of proof of Theorem 1.

Remark 2 Nonlinearity N in the statement of Theorem 1 does not have to be differentiable everywhere. Our proof of the key underlying result, viz. Lemma 1, relied on the fact that it be differentiable at two points only. It may very well be true that N does not need to be differentiable at all. ■

Theorem 2 $\mathcal{M}$ is KS-positivity preserving for an $N \in \mathcal{N}_{inc}$ if and only if

$$\text{skew}(\frac{\partial N(\zeta)}{\partial \zeta}) = 0 \; \forall \zeta \in \mathbb{R}.$$

Proof: Given a $v \in \mathbb{R}^n$, let $\mathcal{N}_v \doteq \{N_v | N_v(u) \doteq N(u+v) - N(v), \; N \in \mathcal{N}_{inc}, \; u \in L_2\}$. Observe that an $N \in \mathcal{N}_v$ satisfies (1) and (3), and furthermore, $N_v(0) = 0$ so that $N_v \in \mathcal{N}$. Note that

$$\langle u - v, M^*(N(u) - N(v)) \rangle = \langle \tilde{u}, M^* N_v(\tilde{u}) \rangle$$

where $\tilde{u} \doteq u - v$. In addition, $\text{skew}(\frac{\partial N_v(\zeta)}{\partial \zeta}) = \text{skew}(\frac{\partial N(\zeta)}{\partial \zeta})$. The result then follows by applying Theorem 1 of [5]. QED.

4 Discussion

Positivity preservation results are of utmost importance in deducing stability of $\mathcal{S}$ using multiplier techniques (see [1, Theorem 1, 2], [4],[5] and references therein for a relevant discussion). Likewise, incremental positivity preservation results are of importance in deducing incremental stability of $\mathcal{S}$ (see [6] for a definition of, and some results on, the incremental stability).

Our preliminary researches seem to indicate that if H is restricted to be a stable, LTI transfer function, then the establishment of stability of the feedback system using $\mathcal{M}$ is equivalent to the establishment of its incremental positivity, although the same can not be said if H is a dynamic nonlinearity. The importance of such a characterization is self-evident. Further work is needed before a concrete characterization can be stated.

5 Conlusion

Zames-Falb multipliers, known to preserve positivity of incrementally positive nonlinearities, may fail to preserve incremental positivity thereof.

References

[1] G. Zames and P. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM J. Control and Optimization*, 6:89–108, 1968.

[2] M. G. Safonov and G. Wyetzner. Computer-aided stability analysis renders Popov criterion obsolete. *IEEE Trans. Autom. Control*, AC-32(12):1128–1131, 1987.

[3] P. B. Gapski and J. C. Geromel. A convex approach to the absolute stability problem. *IEEE Trans. Autom. Control*, AC-39(9):1929–1932, September 1994.

[4] F. D'Amato, A. Megretski, U. Jönsson, and M. Rotea. Integral quadratic constraints for monotonic and slope-restricted diagonal operators. In *Proc. American Control Conf.*, pages 2375–2379, San Diego, CA, June 2–4, 1999. IEEE Press, New York.

[5] M.G. Safonov and V.V. Kulkarni. Zames-Falb multipliers for MIMO nonlinearities. In *Proc. American Control Conf.*, pages 4144–4148, Chicago, IL, June 28–30, 2000. IEEE Press, New York. ftp://routh.usc.edu/pub/safonov/safo00a.pdf.

[6] G. Zames. On the input–output stability of time-varying nonlinear feedback systems — Part I: Conditions derived using concepts of loop gain, conicity, and positivity. *IEEE Trans. Autom. Control*, AC-11(2):228–238, April 1966.