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Multiplier IQC's for Uncertain Time-delays*

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Abstract

This paper describes a set of delay-dependent IQC's for time-delay uncertainty. The set is linearly parameterized in terms of the frequency-response of a complex scalar-valued multiplier. Using LMI optimization techniques, one may compute optimal multipliers and thereby obtain less conservative IQC stability robustness bounds for systems with uncertain time-delays.

1 Introduction

Stability criteria for time-delay systems tend to fall into one of two categories according to their dependence upon delay size: *delay-dependent* or *delay-independent*. Delay-independent criteria provide conditions for stability without regard for the size of the time delays. They tend to be more conservative than delay-dependent criteria which may exploit prior knowledge of upper-bounds on the amount of time-delay.

The robust stability methodology is useful in dealing with structured uncertainties (see [2],[4]). Time-delays can be considered as structured uncertainties and time-delay systems can be analyzed using these robust control theories[11]. Many of methods that have been developed within the area of robust control during the last decade have been shown to be reformulated to fall within the framework of the integral quadratic constraints (IQC's)[9]. Fu *et al.*[5] and Jun *et al.*[7] provided delay-dependent results for robust stability using this IQC approach and the linear matrix inequalities (LMI's) technique, which give an estimate of the maximum time-delay which preserves robust stability.

Some recent papers on time-delay systems, like [3] and [8], derive sufficient conditions for stability in the form of LMI using Lyapunov functionals. Others have used input-output stability theories such as the small-gain criterion [13] and its generalizations, like the IQC stability analysis method [9]. The input-output methods seem to offer advantages over Lyapunov methods in facilitating the decomposition of the stability robustness analysis problem for complex systems having several sorts of uncertain subsystems into subproblems of finding an IQC for each of the subsystems — e.g., for each uncertain time-delay and each nonlinearity or other uncertainty. Once the subsystem IQC's are in hand, stability analysis for the composite system

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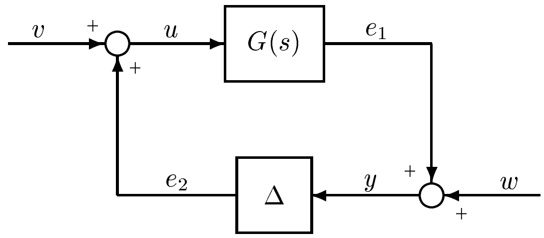

Figure 1: Basic feedback configuration

is then a relatively straightforward matter of optimizing IQC scalings (and, sometimes, other free parameters like Popov multipliers) in an effort to identify a single aggregate IQC for the system.

This paper considers robust stability analysis of systems with time delay based on an IQC approach. It also provides a proper parameterization of Δ, which Megretski *et al.*[9] did not provide. It is organized as follows: Notation and preliminary background are provided in Section 2. The problem formulation is given in Section 3. The main result is derived in Section 4 where it is shown that the class of known delay-dependent IQC's for time delays can be generalized to a larger linearly-parameterized class. Discussion, geometrical interpretation and comparison with other results are in Section 6. Finally, results are summarized and conclusions are stated in Section 7.

2 Background and Preliminaries

Consider the feedback system in Figure 1 where G and Δ are bounded causal operators on $\mathcal{L}_{2e}^m[0,\infty)$ and $\mathcal{L}_{2e}^l[0,\infty)$, respectively.

Definition 1 *(cf. [9, 12]) The interconnection G and Δ is said to be* well-posed *if the map $(y,u) \mapsto (v,w)$ has a causal inverse on $\mathcal{L}_{2e}^{l+m}[0,\infty)$. The feedback system is said to be* stable *if it is well-posed and inputs $v \in \mathcal{L}_2^m[0,\infty), w \in \mathcal{L}_2^l[0,\infty)$ lead to outputs $e_1, y \in \mathcal{L}_2^l[0,\infty)$ and $e_2, u \in \mathcal{L}_2^m[0,\infty)$. If, in addition, there exists a constant $C > 0$ such that*

$$\int_0^T (|y|^2 + |u|^2)dt \le C \int_0^T (|v|^2 + |w|^2)dt, \quad \forall\, T \ge 0, \tag{1}$$

then, the system is said to be stable with finite gain.

Definition 2 *[9, pp.820] Let $\Pi : j\mathbb{R} \mapsto \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable Hermitian valued function. A bounded operator $\Delta : \mathcal{L}_{2e}^l[0,\infty) \mapsto \mathcal{L}_{2e}^m[0,\infty)$ is said to* satisfy the IQC defined by Π, *if*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\Delta}(y)(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\Delta}(y)(j\omega) \end{bmatrix} d\omega \ge 0 \tag{2}$$

for all $y \in \mathcal{L}_2^l[0,\infty)$.

Table 1: Notation

Symbol	Meaning
$\mathbb{R}$	Set of all real numbers
$\mathbb{R}_+$	Set of positive real numbers
$\mathbb{C}$	Set of all complex numbers
$A(s)^*$	$A(-s)^T$, conjugate transpose
A^{-*}	$= (A^*)^{-1}$
I_q	$q \times q$ identity matrix
$\circ$	Convolution operator
$\mathrm{herm}(m)$	$= \frac{1}{2}(m + m^*)$
$\mathrm{skew}(m)$	$= \frac{1}{2}(m - m^*)$
$\mathrm{diag}(A_1, \cdots, A_n)$	Block diagonal matrix with $A_1, \cdots, A_n$ in diagonal
$\Re e(\cdot)$	Real part of $(\cdot)$
$\Im m(\cdot)$	Imaginary part of $(\cdot)$
$\hat{x}(j\omega)$	Fourier transform of the signal $x(t)$
$< x, y >$	$= \int_{-\infty}^{\infty} y(t)^T x(t) dt$
	$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(j\omega)^* \hat{x}(j\omega) d\omega$
$\|x\|$	$= \sqrt{< x, x >}$

Theorem (IQC Theorem) *(cf. [6, 9, Theorem 1]) Assume that:*

i) for every $\alpha \in [0, 1]$, the interconnection of G and Δ_α is well-posed where Δ_α is a parameterization of Δ which satisfies

a) $\Delta = \Delta_\alpha\big|_{\alpha=1}$,

b) Δ_α is bounded and causal for $\alpha \in [0, 1]$,

c) there exists $\gamma > 0$ such that

$$\|\Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y)\| \leq \gamma |\alpha_1 - \alpha_2| \cdot \|y\| \tag{3}$$

for all $\alpha_1, \alpha_2 \in [0, 1]$,

ii) the interconnection of G and $\Delta_\alpha\big|_{\alpha=0}$ is stable with finite gain,

iii) for every $\alpha \in [0, 1]$, the IQC defined by Π is satisfied by Δ_α, that is,

$$\left\langle \Pi \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix}, \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix} \right\rangle \geq 0, \tag{4}$$

iv) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \quad \forall\, \omega \in \mathbb{R}. \tag{5}$$

Then, the feedback interconnection of G and Δ is stable with finite gain for all $\alpha \in [0, 1]$.

The values $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ and $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ of $\Pi(j\omega)$ represent the small gain theorem and the positivity theorem. The positivity theorem with multiplier can be reformulated with IQC defined by

$$\Pi(j\omega) = \begin{bmatrix} 0 & M^*(j\omega) \\ M(j\omega) & 0 \end{bmatrix}. \tag{6}$$

As mentioned, the input-output stability analysis approach facilitates the decomposition of the stability robustness analysis problem for complex systems with several sorts of uncertain subsystems into subproblems of finding an IQC for each of the subsystems — e.g., for each uncertain time-delay and each nonlinearity or other uncertainty. Once the subsystem IQC's are in hand, stability analysis for the composite system is then a relatively straightforward matter of optimizing IQC scalings in an effort to identify a single aggregate IQC for the system. For example, assume that a overall system Δ consists of n subsystems $\Delta_i, \quad i = 1, \cdots, n$, i.e., $\Delta = \mathrm{diag}\big(\Delta_1, \cdots, \Delta_n\big)$ and that each subsystems, Δ_i, satisfies the IQC defined by

$$\Pi_i = \begin{bmatrix} \Pi_{i(11)} & \Pi_{i(12)} \\ \Pi^*_{i(12)} & \Pi_{i(22)} \end{bmatrix},$$

where the block structures are consistent with the size of the subsystem Δ_i. Then the overall system Δ satisfies the IQC defined by

$$\Pi = \left[\begin{array}{ccc|ccc} \Pi_{1(11)} & & & \Pi_{1(12)} & & \\ & \ddots & & & \ddots & \\ & & \Pi_{n(11)} & & & \Pi_{n(12)} \\ \hline \Pi^*_{1(12)} & & & \Pi_{1(22)} & & \\ & \ddots & & & \ddots & \\ & & \Pi^*_{n(12)} & & & \Pi_{n(22)} \end{array}\right]. \tag{7}$$

Key to successful application of IQC stability methods is the discovery of suitable class of Π_i's for various classes of $\Delta_i(s)$. In the remainder of this paper, we focus on finding a broad class of linearly parameterized Π's for uncertain time-delay Δ.

3 Problem Formulation

Consider a repeated time-delay uncertainty with upper bound $\Delta(s) \triangleq \delta(s) I_q$, where $\delta(s) = e^{-s\tau}$ and $0 \le \tau \le \bar{\tau}$. In this paper, we provide a class of Π's for a repeated time delay uncertainty with an appropriate multiplier $M(j\omega)$ and transformation $S(j\omega)$. Using the Eq. (7), this can be extended to the non-repeated case as well as to systems with other sorts of uncertainties and nonlinearities.

Problem 1 *Given a time-delay uncertainty $\Delta(s) = \delta(s) I_q$ where $\delta(s) = e^{-s\tau}$ and $0 \le \tau \le \bar{\tau}$, find a class of Π's which define the IQC (2) satisfied by $\Delta(s)$.*

4 Main Results

We define $\tilde{y}(j\omega)$ and $\tilde{u}(j\omega)$ to be $\begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix} \triangleq S(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}$ with appropriate invertible transformation matrix $S(j\omega)$. Since $S(j\omega)$ is invertible, there is one-to-one correspondence between

$\begin{bmatrix}\tilde{y}(j\omega)\\ \tilde{u}(j\omega)\end{bmatrix}$ and $\begin{bmatrix}y(j\omega)\\ u(j\omega)\end{bmatrix}$ at each ω. We also define $\tilde{\delta}(j\omega) \stackrel{\triangle}{=} \tilde{u}_i(j\omega)/\tilde{y}_i(j\omega),\ i = 1, \cdots, q$ where $\tilde{y}_i(j\omega)$ and $\tilde{u}_i(j\omega)$ are i^{th} elements of $\tilde{y}(j\omega)$ and $\tilde{u}(j\omega)$, respectively. We will use the symbol ω_* to denote $\omega_* \stackrel{\triangle}{=} \frac{\bar{\tau}\omega}{2}$. The Lemma 1 talks about a transformation $S(j\omega)$ which maps an arc in complex plane to the positive real line.

Lemma 1 *Let the transformation matrix* $S(j\omega)$ *be*

$$S(j\omega) \stackrel{\triangle}{=} \begin{bmatrix}-\cos\omega_* + j\sin\omega_* & \cos\omega_* + j\sin\omega_*\\ \sin\omega_* & -\sin\omega_*\end{bmatrix}. \tag{8}$$

Then, $\Delta(j\omega) = \{\ \Delta \mid\ \Delta = e^{-j\varphi},\ \varphi \in (0, 2\omega_*)\}$ *and* $\omega_* \in (0,\pi)$ *if and only if* $\tilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$.

Proof: *(only if)*

$$\begin{aligned}\tilde{\Delta}(j\omega) &= \frac{\sin\omega_* - \sin\omega_*\Delta(j\omega)}{(-\cos\omega_* + j\sin\omega_*) + (\cos\omega_* + j\sin\omega_*)\Delta(j\omega)}\\ &= \frac{1}{-\cot\omega_* + \dfrac{j + j\Delta(j\omega)}{1-\Delta(j\omega)}}\\ &= \frac{1}{-\cot\omega_* + \dfrac{\sin\varphi + j(1+\cos\varphi)}{1-\cos\varphi + j\sin\varphi}}\\ &= \frac{1}{-\cot\omega_* + \cot\frac{\varphi}{2}}.\end{aligned}$$

We can see that $\tilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$ since $\varphi/2 < \omega_*$ and $-\cot\omega_*$ is strictly increasing when $\omega_* \in (0,\pi)$.

(if) It follows from the fact that $S(j\omega)$ is invertible for $\forall\ \omega_* \in (0,\pi)$ and $\tilde{\Delta}(j\omega)$ has all values in $\mathbb{R}_+ \cup \{0\}$. ■

Next corollary stipulates that the sector transformation matrix can be extended for *repeated* time delays, having upper bound $\bar{\tau}$.

Corollary 1 *Suppose that* $\Delta(j\omega) \stackrel{\triangle}{=} \delta(j\omega)I_q = e^{-j\omega\tau}I_q,\ 0 \le \tau \le \bar{\tau}$ *and* $0 < \omega < \frac{2\pi}{\bar{\tau}}$. *If we define*

$$S(j\omega) \stackrel{\triangle}{=} \begin{bmatrix}-e^{-j\omega_*}I_q & e^{j\omega_*}I_q\\ \sin\omega_* I_q & -\sin\omega_* I_q\end{bmatrix}, \tag{9}$$

then

$$\widetilde{\Delta}(j\omega) \stackrel{\triangle}{=} \widetilde{\delta}(j\omega)I_q = \frac{1}{-\cot\omega_* + \cot(\omega\tau/2)} \cdot I_q. \tag{10}$$

Proof: Since $\begin{bmatrix}\tilde{y}(j\omega)\\ \tilde{u}(j\omega)\end{bmatrix} = S(j\omega)\begin{bmatrix}y(j\omega)\\ u(j\omega)\end{bmatrix}$, we can see that

$$\begin{aligned}\tilde{y}_i(j\omega) &= -e^{-j\omega_*}\cdot y_i(j\omega) + e^{j\omega_*}\cdot u_i(j\omega),\\ \tilde{u}_i(j\omega) &= \sin\omega_*\cdot y_i(j\omega) - \sin\omega_*\cdot u_i(j\omega)\end{aligned}$$

for all $i = 1, \cdots, q$. Applying Lemma 1 leads to

$$\widetilde{\delta}(j\omega) = \frac{\tilde{u}_i(j\omega)}{\tilde{y}_i(j\omega)} = \frac{1}{-\cot\omega_* + \cot(\omega\tau/2)},$$

thus,

$$\widetilde{\Delta}(j\omega) = \widetilde{\delta}(j\omega) I_q = \frac{1}{-\cot\omega_* + \cot(\omega\tau/2)} \cdot I_q$$

since $\tilde{y}(j\omega) = \widetilde{\Delta}(j\omega)\tilde{u}(j\omega)$. ∎

Now we state our main theorem which provides a linearly parameterized set of Π's for repeated time delay uncertainties.

Theorem 1 (Main Theorem) *Let $\Delta(j\omega) \triangleq \delta(j\omega) I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \le \tau \le \bar{\tau}$. Then, $\Delta(j\omega)$ satisfies the IQC (2) defined by*

$$\Pi_M(j\omega) \triangleq \begin{cases} herm\left(\begin{bmatrix} I_q \\ -I_q \end{bmatrix} M(j\omega) \begin{bmatrix} -e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q \end{bmatrix}\right), & |\omega| < \frac{2\pi}{\bar{\tau}} \\ herm\left(\begin{bmatrix} M(j\omega) & 0 \\ 0 & -M(j\omega) \end{bmatrix}\right), & |\omega| \ge \frac{2\pi}{\bar{\tau}} \end{cases} \tag{11}$$

where $M(j\omega) \in \mathbb{C}^{q\times q}$ and $herm(M(j\omega)) \ge 0$ for all $|\omega| < \frac{2\pi}{\bar{\tau}}$.

Proof: It is sufficient to show that

$$\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} \ge 0, \qquad \forall\, \omega \in \mathbb{R}.$$

First, when $\omega = 0$, we have

$$\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} = \begin{bmatrix} I_q \\ I_q \end{bmatrix}^* \text{herm}\left(\begin{bmatrix} -M(j\omega) & M(j\omega) \\ M(j\omega) & -M(j\omega) \end{bmatrix}\right) \begin{bmatrix} I_q \\ I_q \end{bmatrix} = 0.$$

Next, when $0 < \omega < \frac{2\pi}{\bar{\tau}}$, it can be noticed that

$$\begin{aligned} \sin\frac{\bar{\tau}\omega}{2} \cdot \Pi_M(j\omega) &= \sin\frac{\bar{\tau}\omega}{2} \cdot \text{herm}\left(\begin{bmatrix} I_q \\ -I_q \end{bmatrix} M(j\omega) \begin{bmatrix} -e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q \end{bmatrix}\right) \\ &= S(j\omega)^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} S(j\omega) \end{aligned}$$

where $S(j\omega)$ is defined by

$$S(j\omega) \triangleq \begin{bmatrix} -e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q \\ \sin\frac{\bar{\tau}\omega}{2} I_q & -\sin\frac{\bar{\tau}\omega}{2} I_q \end{bmatrix}. \tag{12}$$

Also we can say by Lemma 1 and Corollary 1 that

$$\begin{bmatrix} I_q \\ \widetilde{\Delta}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} I_q \\ \widetilde{\Delta}(j\omega) \end{bmatrix} = \frac{1}{-\cot(\omega\bar{\tau}/2) + \cot(\omega\tau/2)} \text{herm}(M(j\omega)) \ge 0 \tag{13}$$

for $0 < \omega < \frac{2\pi}{\bar{\tau}}$ since $\mathrm{herm}(M(j\omega)) \geq 0$ by assumption.

This leads to

$$\begin{aligned} 0 \leq \begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix} &= \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}^* S(j\omega)^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} S(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix} \\ &= \sin \frac{\bar{\tau}\omega}{2} \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}, \end{aligned}$$

which proves the theorem when $0 < \omega < \frac{2\pi}{\bar{\tau}}$ since $\sin(\bar{\tau}\omega/2) > 0$ when $0 < \omega < \frac{2\pi}{\bar{\tau}}$.

Now, let us consider the case when $\omega \geq \frac{2\pi}{\bar{\tau}}$. We have

$$\begin{aligned} &\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} = \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} \mathrm{herm}(M(j\omega)) & 0 \\ 0 & -\mathrm{herm}(M(j\omega)) \end{bmatrix} \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} \\ &= \mathrm{herm}(M(j\omega)) - \Delta(j\omega)^* \mathrm{herm}(M(j\omega))\Delta(j\omega) = \mathrm{herm}(M(j\omega))\big(I_q - \Delta(j\omega)^*\Delta(j\omega)\big) \qquad (14) \\ &= 0. \end{aligned}$$

It can be proved similarly when $\omega < 0$. ■

5 Parameterization Δ_α and Stability Condition

It can be noticed that the IQC defined by Π_M in Eq. (11) is not satisfied by $\alpha\Delta$ for every $\alpha \in [0, 1]$ in general, that is, a simple *linear* parameterization of Δ, $\Delta_\alpha = \alpha\Delta$, does not satisfy the condition c) in the IQC Theorem. So, another parameterization must be found that enables us to apply the IQC Theorem for stability criterion of time-delay systems. In [9], authors overlooked this critical issue. They did not validate their IQC for time-delay by not providing proper parameterization Δ_α. In this section, such parameterization of Δ for time-delay uncertainty is provided and stability condition of a system with time delay uncertainty is given.

The requirements that the parameterization Δ_α must satisfy are as follows:

a) $\Delta = \Delta_\alpha\big|_{\alpha=1}$,

b) Δ_α is bounded and causal for $\alpha \in [0, 1]$,

c) there exists $\gamma > 0$ such that

$$\big\|\Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y)\big\| \leq \gamma|\alpha_1 - \alpha_2| \cdot \|y\| \qquad (15)$$

for all $\alpha_1, \alpha_2 \in [0, 1]$.

The first plausible candidate is $\Delta_\alpha(j\omega) = e^{-j\omega\tau\alpha} I_q$. This parameterization seems to be reasonable since it can provide the rationale that the feedback system can be said to be stable for all time delay $0 \leq \tau \leq \tau_m$ when the IQC defined by $\Pi_M(j\omega)$ is satisfied by $\Delta(j\omega) = e^{-j\omega\tau_m} I_q$. However, it does not satisfy the condition (15) since $\big\|\Delta_{\alpha_1} - \Delta_{\alpha_2}\big\|_\infty = 2$. Thus, some trick is needed in order to find an appropriate parameterization and apply the IQC theorem with $\Pi_M(j\omega)$.

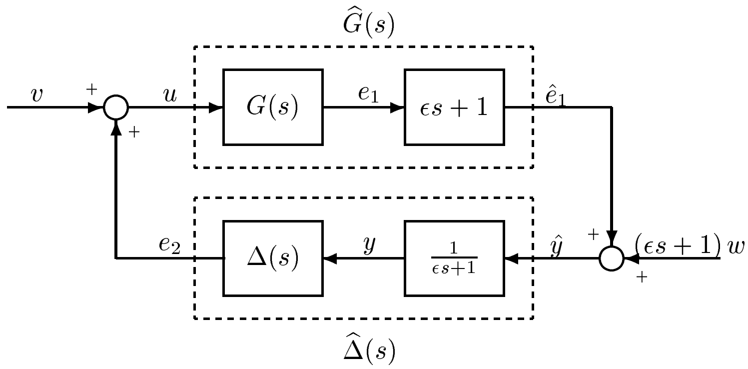

Figure 2: Transformation of the original feedback system for a parameterization of Δ

Before describing the parameterization of Δ, the original feedback interconnection is transformed to the feedback system in Figure 2. Let $\widehat{G}(s) \triangleq (\epsilon s+1)G(s)$ and $\widehat{\Delta}(s) \triangleq \frac{1}{\epsilon s+1}\Delta(s)$. Since $\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} \geq 0$ by Theorem 1, it can be said that

$$\begin{aligned} &\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} I_q & 0 \\ 0 & \frac{1}{\epsilon j\omega+1}I_q \end{bmatrix}^* \begin{bmatrix} I_q & 0 \\ 0 & \frac{1}{\epsilon j\omega+1}I_q \end{bmatrix}^{-*} \Pi_M(j\omega) \begin{bmatrix} I_q & 0 \\ 0 & \frac{1}{\epsilon j\omega+1}I_q \end{bmatrix}^{-1} \begin{bmatrix} I_q & 0 \\ 0 & \frac{1}{\epsilon j\omega+1}I_q \end{bmatrix} \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} \geq 0 \\ \Longleftrightarrow \quad & \begin{bmatrix} I_q \\ \widehat{\Delta}(j\omega) \end{bmatrix}^* \begin{bmatrix} I_q & 0 \\ 0 & \frac{1}{\epsilon j\omega+1}I_q \end{bmatrix}^{-*} \Pi_M(j\omega) \begin{bmatrix} I_q & 0 \\ 0 & \frac{1}{\epsilon j\omega+1}I_q \end{bmatrix}^{-1} \begin{bmatrix} I_q \\ \widehat{\Delta}(j\omega) \end{bmatrix} \geq 0. \end{aligned} \tag{16}$$

Let $\widehat{\Pi}(j\omega) \triangleq \begin{bmatrix} I_q & 0 \\ 0 & (-\epsilon j\omega+1)I_q \end{bmatrix} \Pi_M(j\omega) \begin{bmatrix} I_q & 0 \\ 0 & (\epsilon j\omega+1)I_q \end{bmatrix}$. Notice that $\widehat{G} \approx G$, $\widetilde{\Delta} \approx \Delta$ and $\widehat{\Pi} \approx \Pi_M$ when $\epsilon \ll 1$. The system in Figure 2 has differentiator in its loop, thus, it is assumed that the nominal plant $G(s)$ is strictly proper and $\dot{w} \in \mathcal{L}_{2e}^l[0,\infty)$.

Theorem 2 *Let $\dot{w} \in \mathcal{L}_{2e}[0,\infty)$. Assume that:*

i) for every $\alpha \in [0,1]$, the interconnection of G and Δ_α is well-posed where Δ_α is a parameterization of Δ which satisfies

a) $\Delta = \Delta_\alpha\big|_{\alpha=1}$,

b) Δ_α is bounded and causal for $\alpha \in [0,1]$,

c) there exists $\gamma > 0$ such that

$$\left\| \frac{1}{\epsilon s+1} \circ (\Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y)) \right\| \leq \gamma |\alpha_1 - \alpha_2| \cdot \|y\| \tag{17}$$

for all $\alpha_1, \alpha_2 \in [0,1]$,

ii) G is strictly proper,

iii) the interconnection of G and $\Delta_\alpha\big|_{\alpha=0}$ is stable with finite gain, *iv) for every $\alpha \in [0,1]$, the IQC defined by Π is satisfied by Δ_α, that is,*

$$\left\langle \Pi \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix}, \begin{bmatrix} y \\ \Delta_\alpha(y) \end{bmatrix} \right\rangle \geq 0, \tag{18}$$

v) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq -\epsilon I, \quad \forall \omega \in \mathbb{R}. \tag{19}$$

Then, the feedback interconnection of G and Δ is stable in the sense that $v \in \mathcal{L}_2^m[0,\infty), w, \dot{w} \in \mathcal{L}_2^l[0,\infty)$ lead to $u, e_2 \in \mathcal{L}_2^m[0,\infty), y, e_1 \in \mathcal{L}_2^l[0,\infty)$.

Proof: See appendix. ■

The next lemma provides a parameterization Δ_α.

Lemma 2 *For each $\epsilon > 0$, the parameterization $\Delta_\alpha(j\omega)$ defined by*

$$\Delta_\alpha(j\omega) = e^{-j\omega\tau\alpha} \tag{20}$$

satisfies conditions a),b) and c) of Theorem 2. Furthermore, the condition iv) in Theorem 2 is satisfied by Π_M with the parameterization (20).

Proof: See appendix. ■

Now we assert the stability criterion of time-delay systems with repeated delays with upper bound $\bar{\tau}$. It is a direct application of Theorem 2 and Lemma 2.

(Stability Criterion for Time-delay Systems) *For a given $\bar{\tau} > 0$ suppose that the time-delay uncertainty is expressed as $\Delta(j\omega) \stackrel{\triangle}{=} \delta(j\omega) I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \leq \tau \leq \bar{\tau}$. Let $\dot{w} \in \mathcal{L}_{2e}^q[0,\infty)$. Assume that:*

i) the feedback system is stable with finite gain when $\tau = 0$,

ii) $G(s)$ is strictly proper,

iii) there exists an $M(j\omega)$ with herm$(M(j\omega)) \geq 0$ for $|\omega| < \frac{2\pi}{\bar{\tau}}$ such that

$$\begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix} < 0, \quad \forall\, \omega \in \mathbb{R}, \tag{21}$$

Then, the feedback system in Figure 1 is stable against the time-delay $0 \leq \tau \leq \bar{\tau}$ in the sense that $v, w, \dot{w} \in \mathcal{L}_2^q[0,\infty)$ lead to $u, y, e_1, e_2 \in \mathcal{L}_2^q[0,\infty)$.

Strict properness of $G(s)$ is added restriction caused by transformation for parameterization. However, it is required only for the channel where the time-delay is concerned and can be relaxed when the IQC defined by $\Pi_M(j\omega)$ is satisfied by $\alpha\Delta$ for all $\alpha \in [0,1]$.

Using the Eq. (7) in conjunction with the above criterion, we see that the stability condition of a system with *non-repeated* time-delay can be stated as follows:

(Stability Condition for Non-repeated Time-delay Systems) *Suppose that there are m non-repeated time-delay uncertainties. Let $\Delta(j\omega) = diag\big(e^{-j\omega\tau_1} I_{q_1}, \cdots, e^{-j\omega\tau_m} I_{q_m}\big)$ where $0 \le \tau_i \le \bar{\tau}_i$, $i = 1, \cdots, m$ and $\sum_{i=1}^{m} q_i = q$. Let $\dot{w} \in \mathcal{L}_{2e}^q[0,\infty)$. Assume that:*

i) the feedback system is stable with finite gain when $\tau_i = 0$, $i = 1, \cdots, m$,

ii) $G(s)$ is strictly proper,

iii) there exists each $M_i(j\omega) \in \mathbb{C}^{q_i \times q_i}$ with $herm(M_i(j\omega)) \ge 0$ for $|\omega| < \frac{2\pi}{\bar{\tau}_i}$, $i = 1, \cdots, m$ respectively such that

$$\begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix}^* \Pi_M^{(aug)}(j\omega) \begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix} < 0, \quad \forall\, \omega \ge 0, \tag{22}$$

where $\Pi_M^{(aug)}(j\omega)$ is augmented as in Eq. (7),

Then, the feedback system in Figure 1 with time-delay uncertainty $\Delta(j\omega)$ is stable against time-delay uncertainties $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, \cdots, m$ in the sense that $v, w, \dot{w} \in \mathcal{L}_2^q[0,\infty)$ lead to $u, y, e_1, e_2 \in \mathcal{L}_2^q[0,\infty)$.

6 Discussion

It can be shown that the IQC's for time-delays introduced by Megretski *et al.* [9, pp.827] and by Scorletti [11] are special cases of the broad class of parameterized IQC's given by our Theorem 1. In this section, we show that each of these special cases corresponds to a particular choice for our multiplier parameter $M(j\omega)$. For simplicity, we assume that $q = 1$.

The inequality used by [9] to find an IQC for time delay $\tau \in [0, \bar{\tau}]$ is

$$\psi_1(\omega_*)(|j\omega_* u(j\omega) + y(j\omega)|^2 - (1+\omega_*^2)|y(j\omega)|^2) \ge \psi_2(\omega_*)|y(j\omega) - u(j\omega)|^2 \tag{23}$$

where $\psi_{1,2}$ are the functions defined by

$$\psi_1(\omega) = \begin{cases} \frac{\sin\omega}{\omega}, & |\omega| \le \pi \\ 0, & |\omega| > \pi \end{cases}, \qquad \psi_2(\omega) = \begin{cases} \cos\omega, & |\omega| \le \pi \\ 0, & |\omega| > \pi \end{cases}.$$

With this inequality, we can derive the $\Pi(j\omega)$ for time-delay uncertainty

$$\Pi = \begin{bmatrix} -\cos\omega_* - \omega_*\sin\omega_* & \cos\omega_* + j\sin\omega_* \\ \cos\omega_* - j\sin\omega_* & -\cos\omega_* + \omega_*\sin\omega_* \end{bmatrix}. \tag{24}$$

Comparing the Eq. (24) with the Eq. (11), we can see that we have the Eq. (24) when $M(j\omega) = 1 + j\omega_*$.

Now let us consider the IQC in [11]. The uncertainty for time delay in [11] is defined as $\Delta(j\omega) = e^{-j\omega\tau} - 1$. Thus, if we reformulate the IQC in [11] with time uncertainty $\Delta(j\omega) = e^{-j\omega\tau}$, then we have

$$\Pi = \begin{bmatrix} -2\cot\omega_* & \cot\omega_* + j \\ \cot\omega_* - j & 0 \end{bmatrix} \tag{25}$$

We can easily see that we can get Eq. (25) from Eq. (11) with $M(j\omega) = \frac{1}{\sin\omega_*}(1 + j\cot\omega_*)$.

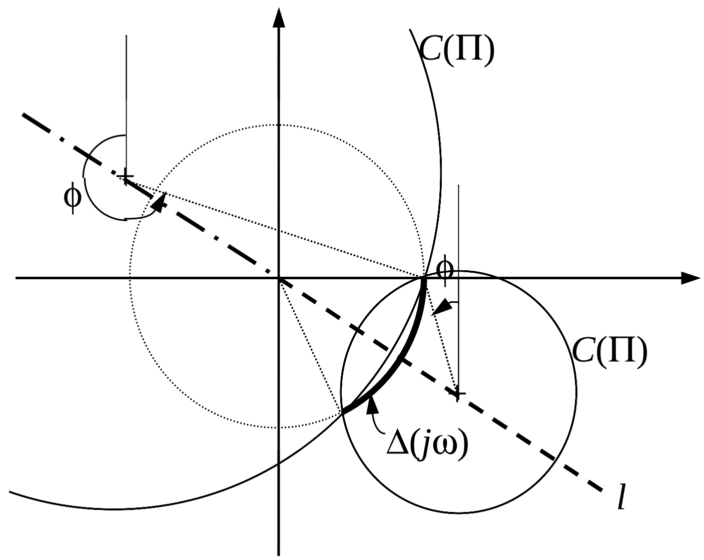

Figure 3: The graph of $\Delta(j\omega)$ and $\mathcal{C}(\Pi(j\omega))$ when $\omega_* \in [0,\pi)$. $\mathcal{C}(\Pi)$ is the circle with center on the line ℓ. The phase angle $\phi \triangleq \measuredangle M(j\omega)$ determines which circle.

Thus, our result includes the results of [9] and [11]. Furthermore, the Eq. (11) is linear with respect to $M(j\omega)$. This ensures that the problem of finding the optimal multiplier $M(j\omega)$ which provides least conservative bound of delay margin is, at each frequency ω, a linear matrix inequality (LMI); consequently, the optimal multiplier $M(j\omega)$ may be readily computed — see, for example, Boyd *et al.* [1] and Safonov *et al.* [10].

Proposition 1 *Suppose* $\Pi_{ii} \in \mathbb{R}, i = 1,2$ *and* $\Pi_{12} \in \mathbb{C}$. *If* $\det \Pi < 0$ *and* $\Pi_{22} \neq 0$, *then the locus* $\Delta(j\omega)$ *which satisfy the quadratic equation*

$$\begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} = 0 \tag{26}$$

is circle $\mathcal{C}(\Pi(j\omega)) = \{\, \Delta \mid |\Delta(j\omega) - c| = r \,\}$ *where* $c = -\frac{\Pi_{12}^*}{\Pi_{22}}$ *and* $r = \sqrt{\frac{|\Pi_{12}|^2}{\Pi_{22}^2} - \frac{\Pi_{11}}{\Pi_{22}}} = \frac{\sqrt{-\det \Pi}}{|\Pi_{22}|}$.

Proof:

$$\begin{aligned} 0 &= \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \Delta(j\omega) \end{bmatrix} \\ &= \Pi_{11} + \Pi_{12}\Delta + \Delta^*\Pi_{12}^* + \Delta^*\Pi_{22}\Delta \\ &= \Pi_{22}\left(\Delta + \Pi_{22}^{-1}\Pi_{12}^*\right)^*\left(\Delta + \Pi_{22}^{-1}\Pi_{12}^*\right) + \Pi_{11} - \Pi_{12}\Pi_{12}^*\Pi_{22}^{-1} \\ &= \Pi_{22}|\Delta + \Pi_{22}^{-1}\Pi_{12}^*|^2 + \Pi_{11} - \Pi_{12}\Pi_{12}^*\Pi_{22}^{-1} \\ &= \Pi_{22}\left(|\Delta - c|^2 - r^2\right) \end{aligned}$$

■

If $\Pi_{22} = 0$ and $\Pi_{12} \neq 0$, the set of $\Delta(j\omega)$'s which satisfy Eq. (26) is the line which can be represented by the equation

$$(\Pi_{12} + \Pi_{12}^*) \cdot \frac{\Delta(j\omega) + \Delta(j\omega)^*}{2} \; + \; (\Pi_{12} - \Pi_{12}^*) \cdot \frac{\Delta(j\omega) - \Delta(j\omega)^*}{2} \; + \; \Pi_{11} \; = \; 0,$$

that is,

$$2\Re e(\Pi_{12}) \cdot \Re e(\Delta(j\omega)) \; - \; 2\Im m(\Pi_{12}) \cdot \Im m(\Delta(j\omega)) \; + \; \Pi_{11} \; = \; 0.$$

By Proposition 1, it is possible to interpret our multiplier-based IQC's for time-delays in terms of a circle that passes through the two end points of the arc $\Delta(j\omega) = \{ \, \Delta \mid \; \Delta = e^{-j\varphi}, \; \varphi \in (0, 2\omega_*) \, \}$, 1 and $e^{-2j\omega_*}$, in the Nyquist plane. The phase angle $\phi \stackrel{\triangle}{=} \measuredangle M(j\omega)$ determines which one of the many circles passing through these two points. See Figure 3. Notice that the centers of circles lie on the line ℓ which passes the origin and slope ω_* regardless of the value of $M(j\omega)$. And we can see that the inequality (4) implies the inside or outside of a circle according to the sign of Π_{22}. When $\Pi_{22} < 0$ it is the inside of a circle by Lemma 1 and outside when $\Pi_{22} > 0$. If the center of a circle lies on the dashed ray in Figure 3, the region which the inequality (4) represents is the inside of the circle. On the other hand when the center lies on the dash-dotted ray, it is the outside. Just one of the many possible circles determined by our multipliers was considered in [9] and [11]. A line can be thought as a circle with infinite radius which occurs when $\phi = \pi/2 - \omega_*$. Our multiplier IQC's for time-delay are less conservative because they use the multiplier $M(j\omega)$ to parameterize the entire class of *all* circles which pass through those two points.

7 Conclusion

Working from the IQC perspective, we provide a broad class of $\Pi(j\omega)$'s for time delays which are linearly parameterized in terms of a complex multiplier $M(j\omega)$ with $\text{herm}(M(j\omega)) > 0$. The multipliers may be readily optimized for each specific application using LMI techniques. It can be applied to non-repeated time-delay as well as repeated time-delay uncertainty. We also provide a proper parameterization which validates our IQC to apply IQC theorem for stability test. The results are less conservative than if a particular multiplier were to be pre-specified. In particular, we have shown that the time-delay IQC's of Megretski *et al.* [9] and of Scorletti [11] correspond to two such particular choices for the multiplier, which means that our multiplier-parameterized IQC's will generally produce better results.

Appendix

Lemma 3 *Consider the feedback systems in Figure 1 and Figure 2. Let $\epsilon > 0$. Assume that G is strictly proper and $\dot{w} \in \mathcal{L}_{2e}[0, \infty)$. Then, if the feedback system in Figure 1 is stable with finite gain, the feedback system in Figure 2 is stable with finite gain.*

Proof: By hypothesis, there exist $c_1 > 0$, $c_2 > 0$, $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$ such that

$$\|u\| \leq c_1 \|v\| + c_2 \|w\|, \qquad \|y\| \leq \tilde{c}_1 \|v\| + \tilde{c}_2 \|w\|. \tag{27}$$

Thus, we have

$$\|\hat{y}\| \leq \epsilon \|\dot{w}\| + \|w\| + \left\|\widehat{G}\right\| \|u\| \leq \epsilon \|\dot{w}\| + \|w\| + \left\|\widehat{G}\right\| \left(c_1 \|v\| + c_2 \|w\|\right).$$

$\|\widehat{G}\|$ is bounded since G is strictly proper and bounded. Therefore, there exist $C > 0$ such that

$$\|u\| + \|\hat{y}\| \leq C\big(\|\dot{w}\| + \|w\| + \|v\|\big).$$

■

Lemma 4 *Consider the feedback systems in Figure 1 and Figure 2. Let $\epsilon > 0$. Assume that G is strictly proper and $\dot{w} \in \mathcal{L}_{2e}[0,\infty)$. Then, if the feedback system in Figure 2 is stable with finite gain, the feedback system in Figure 1 is stable in the sense that $v, w, \dot{w} \in \mathcal{L}_2[0,\infty)$ lead to $u, y, e_1, e_2 \in \mathcal{L}_2[0,\infty)$.*

Proof: By hypothesis, there exists a constant $C > 0$ such that

$$\int_0^T (|\hat{y}|^2 + |u|^2)dt \leq C \int_0^T (|v|^2 + |\epsilon\dot{w} + w|^2)dt, \quad \forall\, T \geq 0.$$

Thus, $v, w, \dot{w} \in \mathcal{L}_2[0,\infty)$ lead to $u, \hat{y} \in \mathcal{L}_2[0,\infty)$. And we have $y \in \mathcal{L}_2[0,\infty)$ since

$$\|y\| = \left\|\frac{1}{\epsilon s + 1}\hat{y}\right\| \leq \left\|\frac{1}{\epsilon s + 1}\right\| \|\hat{y}\| < \infty.$$

Finally, we get $e_1, e_2 \in \mathcal{L}_2[0,\infty)$ easily from the fact that $v, w, \dot{w}, u, y \in \mathcal{L}_2[0,\infty)$ and $e_1 + w = y$, $e_2 + v = u$. ■

Proof of Theorem 2: Consider the system in Figure 2. Let $\widehat{G} \triangleq G \circ (\epsilon s + 1)I$, $\widehat{\Delta} \triangleq \frac{1}{\epsilon s+1} I \circ \Delta$ and $\widehat{\Delta}_\alpha \triangleq \frac{1}{\epsilon s+1} I \circ \Delta_\alpha$. Also define $\widehat{\Pi}(j\omega)$ to be $\begin{bmatrix} I & 0 \\ 0 & (-\epsilon j\omega + 1)I \end{bmatrix} \Pi(j\omega) \begin{bmatrix} I & 0 \\ 0 & (\epsilon j\omega + 1)I \end{bmatrix}$. Then it can be noticed from the hypothesis that the parameterization $\widehat{\Delta}_\alpha$ satisfies the conditions a), b) and c) in the IQC Theorem. The condition i) in the IQC Theorem is satisfied by hypothesis i). The condition ii) in the IQC Theorem is satisfied by hypothesis ii), iii) and Lemma 3. The condition iii) in the IQC Theorem is satisfied since

$$\begin{aligned}
& \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0 \\
\Longleftrightarrow \; & \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\epsilon j\omega+1} I \end{bmatrix}^* \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\epsilon j\omega+1} I \end{bmatrix}^{-*} \Pi(j\omega) \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\epsilon j\omega+1} I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\epsilon j\omega+1} I \end{bmatrix} \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0 \\
\Longleftrightarrow \; & \begin{bmatrix} I \\ \widehat{\Delta}(j\omega) \end{bmatrix}^* \widehat{\Pi}(j\omega) \begin{bmatrix} I \\ \widehat{\Delta}(j\omega) \end{bmatrix} \geq 0.
\end{aligned}$$

The condition iv) in the IQC Theorem is satisfied since

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in \mathbb{R},$$

$$\Longleftrightarrow \quad \left|\epsilon j\omega + 1\right|^2 \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in \mathbb{R}$$

$$\Longleftrightarrow \quad \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \frac{1}{\epsilon j\omega+1} I & 0 \\ 0 & \frac{1}{\epsilon j\omega+1} I \end{bmatrix}^{-*} \Pi(j\omega) \begin{bmatrix} \frac{1}{\epsilon j\omega+1} I & 0 \\ 0 & \frac{1}{\epsilon j\omega+1} I \end{bmatrix}^{-1} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in \mathbb{R}$$

$$\Longleftrightarrow \quad \begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \frac{1}{\epsilon j\omega+1} I & 0 \\ 0 & I \end{bmatrix}^{-*} \widehat{\Pi}(j\omega) \begin{bmatrix} \frac{1}{\epsilon j\omega+1} I & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in \mathbb{R},$$

$$\Longleftrightarrow \quad \begin{bmatrix} \widehat{G}(j\omega) \\ I \end{bmatrix}^* \widehat{\Pi}(j\omega) \begin{bmatrix} \widehat{G}(j\omega) \\ I \end{bmatrix} < 0, \quad \forall\, \omega \in \mathbb{R}.$$

Therefore, by the IQC Theorem, the feedback interconnection of $\widehat{G}$ and $\widehat{\Delta}$ is stable with finite gain, which completes the proof by Lemma 4. ■

Proof of Lemma 2: It is trivial to check that the conditions a) and b) hold with the parameterization (20). Let us check the condition c).

If $\alpha_1 = \alpha_2$, then any $\gamma > 0$ satisfies the condition c). So, let us consider the case when $\alpha_1 \neq \alpha_2$. Define $\widehat{\Delta}_\alpha(s)$ to be $\widehat{\Delta}_\alpha(s) \triangleq \frac{1}{\epsilon s+1}\Delta_\alpha(s)$. Then, we have

$$\begin{aligned}
\left\|\widehat{\Delta}_{\alpha_1} - \widehat{\Delta}_{\alpha_2}\right\|_\infty &= \left\| \frac{\widehat{\Delta}_{\alpha_1} - \widehat{\Delta}_{\alpha_2}}{\alpha_1 - \alpha_2} \cdot (\alpha_1 - \alpha_2) \right\|_\infty \leq \left\| \frac{e^{-j\omega\tau\alpha_1} - e^{-j\omega\tau_1\alpha_2}}{(\epsilon j\omega + 1)(\alpha_1 - \alpha_2)} \right\|_\infty \cdot |\alpha_1 - \alpha_2| \\
&= |\alpha_1 - \alpha_2| \cdot \sup_{\omega\in\mathbb{R}} \left| \frac{e^{-j\omega\tau\alpha_1} - e^{-j\omega\tau\alpha_2}}{(\epsilon j\omega + 1)(\alpha_1 - \alpha_2)} \right| = |\alpha_1 - \alpha_2| \cdot \sup_{\omega\in\mathbb{R}} \sqrt{\frac{4\sin^2(\omega\tau(\alpha_1 - \alpha_2)/2)}{(\epsilon^2\omega^2 + 1)(\alpha_1 - \alpha_2)^2}} \\
&\leq |\alpha_1 - \alpha_2| \cdot \sup_{\omega\in\mathbb{R}} \sqrt{\frac{4\sin^2(\omega\bar{\tau}(\alpha_1 - \alpha_2)/2)}{(\epsilon^2\omega^2 + 1)(\alpha_1 - \alpha_2)^2}} \\
&\leq |\alpha_1 - \alpha_2| \cdot \sup_{x\in\mathbb{R}} \frac{\bar{\tau}}{\epsilon} \left| \frac{\sin x}{x} \right| \\
&\leq \frac{\bar{\tau}}{\epsilon} |\alpha_1 - \alpha_2|
\end{aligned}$$

for $\alpha_1, \alpha_2 \in [0, 1]$. The substitution $x = \frac{\omega\bar{\tau}(\alpha_1 - \alpha_2)}{2}$ was applied in fifth step. Thus, there exists $\gamma > 0$ such that

$$\left\|\widehat{\Delta}_{\alpha_1}(y) - \widehat{\Delta}_{\alpha_2}(y)\right\| \leq \gamma|\alpha_1 - \alpha_2| \cdot \|y\|$$

for all $\alpha_1, \alpha_2 \in [0, 1]$ since

$$\frac{\bar{\tau}}{\epsilon}|\alpha_1 - \alpha_2| \geq \left\|\widehat{\Delta}_{\alpha_1} - \widehat{\Delta}_{\alpha_2}\right\|_\infty = \sup_{\|y\|\neq 0} \frac{\left\|\widehat{\Delta}_{\alpha_1}(y) - \widehat{\Delta}_{\alpha_2}(y)\right\|}{\|y\|} \geq \frac{\left\|\widehat{\Delta}_{\alpha_1}(y) - \widehat{\Delta}_{\alpha_2}(y)\right\|}{\|y\|}.$$

Now let us check the condition iv) in Theorem 2. Let $\widetilde{\Delta}_\alpha(j\omega)$ be

$$\widetilde{\Delta}_\alpha(j\omega) = \frac{1}{-\cot(\omega\bar{\tau}/2) + \cot(\omega\tau\alpha/2)}.$$

It is trivial to check the condition iv) when $\omega = 0$. Consider the case when $0 < \omega < \frac{2\pi}{\bar{\tau}}$. In this frequency range, it can be said from Lemma 1 that

$$\begin{aligned} &\begin{bmatrix} I_q \\ \Delta_\alpha(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} I_q \\ \Delta_\alpha(j\omega) \end{bmatrix} \geq 0, \quad \forall\, \alpha \in [0,1] \\ \Longleftrightarrow &\begin{bmatrix} I_q \\ \widetilde{\Delta}_\alpha(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} I_q \\ \widetilde{\Delta}_\alpha(j\omega) \end{bmatrix} \geq 0, \quad \forall\, \alpha \in [0,1] \end{aligned} \tag{28}$$

Therefore, from Eq. (13), we have

$$\begin{bmatrix} I_q \\ \widetilde{\Delta}_\alpha(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M(j\omega)^* \\ M(j\omega) & 0 \end{bmatrix} \begin{bmatrix} I_q \\ \widetilde{\Delta}_\alpha(j\omega) \end{bmatrix} = \frac{1}{-\cot(\omega\bar{\tau}/2) + \cot(\omega\tau\alpha/2)} \mathrm{herm}(M(j\omega)) \geq 0$$

for every $\alpha \in [0,1]$, which proves the Eq. (28). When $\omega \geq \frac{2\pi}{\bar{\tau}}$, from Eq. (14), we have

$$\begin{bmatrix} I_q \\ \Delta_\alpha(j\omega) \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} I_q \\ \Delta_\alpha(j\omega) \end{bmatrix} = \mathrm{herm}(M(j\omega))\big(I_q - \Delta_\alpha(j\omega)^*\Delta_\alpha(j\omega)\big) = 0$$

for every $\alpha \in [0,1]$. Thus, with same procedure in the proof of Theorem 1, we can see that the condition iv) in Theorem 2 is satisfied by Π_M in Eq. (11) with the parameterization (20). ■

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