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# All Multipliers for Repeated Monotone Nonlinearities

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**Abstract**

We have derived the largest class of bounded linear time-invariant operators that preserve positivity of repeated monotone nonlinearities. It reduces conservativeness of IQC stability analysis of systems having repeated monotone nonlinearities to the minimum possible. Utility of this characterization stems from the fact that such nonlinearities are typically encountered in stability of anti-reset windup schemes among other applications. We have also derived the time varying counterpart of this class of operators.

**Keywords**

repeated nonlinearities, stability, integral quadratic constraint (IQC).

## I. Introduction

In stability analysis, a given system $\mathcal{S}$ is often decomposed into two interconnected subsystems — a linear time invariant subsystem $H$ in the feedforward path and an otherwise subsystem $\Delta$ in the feedback path. More often than not, a repeated monotone nonlinearity, say $N$, is encountered as the subsystem $\Delta$ (see, e.g., [1], [2], [3], [4], [5] and references therein). A key step in multiplier based stability analysis of such systems is to characterize a class of *multipliers*, i.e. a class of *convolution operators*, such that every element $M$ of it *preserves positivity* of $N$ in the sense that $N \geq 0$ implies $M^*N \geq 0$. Stability of the system is then deduced if there exists *at least one* such multiplier $M$ such that $MH > 0$ and if, in addition, $H$ has a finite gain (see [6, Theorem 2], [1], [7] and references therein for a detailed relevant discussion). Effectively, positivity preserving multipliers give an integral quadratic constraint (IQC) characterization of $N$ (see [8] for IQC's — theory and applications). The *larger* the class of the positivity preserving multipliers the *better* it is, for the sharper is its IQC characterization and the lesser is the conservativeness in the stability analysis.

The best available class of positivity preserving multipliers so far for repeated SISO monotone nonlinearities is the one recently derived by D'Amato et al [1]. Whether it is the best *possible* as well has been unclear. It turns out that they have stipulated an unnecessary condition on the multipliers. Identifying and relaxing this condition, in this note we have obtained a larger — indeed, the largest possible — class of positivity preserving multipliers for such nonlinearities. Specifically, we have characterized the largest possible classes of both linear time-invariant as well as *linear time varying* operators that preserve positivity of such nonlinearities. Essentially, our results generalize the non-repeated monotone nonlinearity results of Willems [9, Ch. 3] to the case of repeated monotone nonlinearities.

Saturation nonlinearities, dead zone nonlinearities, sigmoidal nonlinearities are some of the many examples of monotone nonlinearities. When input-output channels of $\Delta$, or a sub block of it, feature the *same* such nonlinearity, an instance of repeated monotone nonlinearity is on hand. Computation of stability margin of anti windup schemes is one of the engineering applications in which repeated monotone nonlinearities appear (see, e.g., [1], [3], [10]). Reduction of conservatism in such stability margin estimates is thus a motivating application of this paper.

The paper is organized as follows. In Section II, the necessary terminology is introduced and the problems are formally posed in Section III. Background results are in Section IV. Our main results are presented in Section V and discussed in Section VI. The paper is concluded in Section VII.

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## II. Preliminaries

The notation used is summarized in Table I. Capital letter symbols, e.g. $F$ and $G$, denote operators whereas small letters, e.g. $x$ and $y$, denote real signals which may possibly be vector-valued or matrix-valued. The vector space $\ell_2^n$ is generally referred to as $\ell_2$. $\mathbb{Z}$ denotes the set of all integers. $(\cdot)^*$ denotes conjugate transpose of a vector or matrix $(\cdot)$; $(\cdot)^T$ denotes its transpose. A sequence $\{x(k)\}_{k=-\infty}^{\infty}$ is described simply as $\{x\}$. Statements of the form "A related to B" and "C related to D" are abbreviated as "A (C) related to B (D)". $D \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz* if each of its eigenvalues has a strictly negative real part. Other terms not defined here may be found in [11] and [8].

TABLE I
Notation

| Symbol | Meaning |
|---|---|
| $\mathbb{R}$ ($\mathbb{C}$) | Set of all real (complex) numbers. |
| $\mathbb{Z}$ | Set of all integers. |
| $\mathrm{herm}(m)$ | $= \frac{1}{2}(m+m^*)$, for $m \in \mathbb{C}^{n\times n}$ or $\mathbb{R}^{n\times n}$. |
| $\mathrm{skew}(m)$ | $= \frac{1}{2}(m-m^*)$, for $m \in \mathbb{C}^{n\times n}$ or $\mathbb{R}^{n\times n}$. |
| $\langle x, y\rangle$ | $= \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} y(k)^T x(k) & \text{for discrete time signals;} \\ \displaystyle\int_{-\infty}^{\infty} y(t)^T x(t)\ dt & \text{for continuous time signals.} \end{cases}$ |
| $\|\|x\|\|$ | $= \sqrt{\langle x, x\rangle}$. |
| $\|\|x\|\|_1$ | $= \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} \|x(k)\| & \text{if } x \text{ discrete time;} \\ \displaystyle\int_{-\infty}^{\infty} \|x(t)\|\ dt & \text{if } x \text{ continuous time.} \end{cases}$ |
| $\ell_2$ | Space of discrete time signals $x$ for which $\|\|x\|\|$ exists. |
| $\mathcal{L}_2$ | Space of continuous time signals $x$ for which $\|\|x\|\|$ exists. |
| $\widehat{x}(\cdot)$ | Fourier transform of $x$, either discrete or continuous. |
| $\delta$ | $= \begin{cases} \text{Kronecker } \delta(k), & \text{if discrete time;} \\ \text{Dirac } \delta(t), & \text{if continuous time.} \end{cases}$ |
| $\lambda_i(H)$ | $i$-th eigenvalue of matrix $H$. |
| $\underline{\lambda}(D)$ | Least eigenvalue of a Hermitian matrix $D$. |
| MIMO | Multi-Input-Multi-Output. |
| SISO | Single-Input-Single-Output. |

*Definition 1:* [operator: positive, bounded]
An operator $F$ mapping a space $X$ into itself is said to be *positive* if $\langle x, Fx\rangle \geq 0\ \forall x \in X$. A set $S$ is said to be *bounded* if there exists a $\gamma \in \mathbb{R}^+$ such that $\|y\| < \gamma$ for all $y \in S$. An operator $F : X \to Y$ is said to be *bounded* if the image under $F$ of every bounded subset of $X$ is a bounded subset of $Y$. □

*Definition 2:* [sequences: similarly ordered, unbiased]
The sequences $\{x\}$ and $\{y\}$ of real scalars are said to be *similarly ordered* if $x(k) < x(l)$ implies $y(k) \leq y(l)$ for all $k, l \in \mathbb{Z}$. They are said to be *unbiased* if $x(k)y(k) \geq 0\ \forall k$. They are said to be *similarly ordered and symmetric* if they are unbiased and, in addition, the sequences $\{|x|\}$ and $\{|y|\}$ are similarly ordered. □

*Definition 3:* [associated matrix, kernel]
Given a bounded possibly time varying linear operator $M : \ell_2^p \to \ell_2^p$, $y = Mx$ is given as

$$y(k) \doteq \sum_{l=-\infty}^{\infty} \overline{m}_{k,l}\ x(l)\ \ \forall k \in \mathbb{Z},$$

where $\overline{m}_{k,l} \in \mathbb{R}^{p\times p} \ \ \forall k,l$; the *associated matrix* $\widetilde{M}$ of $M$ is defined as

$$\widetilde{M} \doteq \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \overline{m}_{-1,-1} & \overline{m}_{-1,0} & \overline{m}_{-1,1} & \overline{m}_{-1,2} & \ddots & \ddots \\ \ddots & \overline{m}_{0,-1} & \overline{m}_{0,0} & \overline{m}_{0,1} & \overline{m}_{0,2} & \ddots & \ddots \\ \ddots & \overline{m}_{1,-1} & \overline{m}_{1,0} & \overline{m}_{1,1} & \overline{m}_{1,2} & \overline{m}_{1,3} & \ddots \\ \ddots & \overline{m}_{2,-1} & \overline{m}_{2,0} & \overline{m}_{2,1} & \overline{m}_{2,2} & \overline{m}_{2,3} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

The symbol $m_{ij},\ \ i,j \in \mathbb{Z}$ denotes the $(i,j)$-th scalar element of the matrix $\widetilde{M}$; for example, $m_{00}$ denotes the upper left entry in the $p\times p$ matrix $\overline{m}_{0,0}$ and $m_{-p,0}$ denotes the upper left entry in the $p \times p$ matrix $\overline{m}_{-1,0}$. If $\overline{m}_{k,l} = \overline{m}_{k+n,l+n} \ \forall k,l,n \in \mathbb{Z}$ then $\widetilde{M}$ is said to be *block Toeplitz* and $M$ is said to be a *time invariant operator* or, alternatively, a *convolution operator*. For a bounded possibly time varying continuous time linear operator $M : \mathcal{L}_2 \to \mathcal{L}_2$

$$y(t) = \int_{-\infty}^{\infty} \overline{m}(t,\tau)x(\tau)\ d\tau \quad \forall t \in \mathbb{R}.$$

the *kernel* $\overline{m}(t,\tau) \in \mathbb{R}^{p\times p}$ is the counterpart of $\overline{m}_{k,l}$. In the continuous time case, $M$ is called a *time invariant operator* or, alternatively, a *convolution operator* if $\overline{m}(t,\tau) = \overline{m}(t+\nu,\tau+\nu)\ \forall t,\tau,\nu \in \mathbb{R}$. For a convolution operator $M$, a shorthand notation for $\overline{m}(t,\tau)$ and $\overline{m}_{i,j}$ is $\overline{m}(t-\tau)$ and $\overline{m}(i-j)$, respectively with $\overline{m}(t)$ and $\overline{m}(k)$ denoting the respective *impulse response*. □

*Definition 4:* [hyperdominance, dominance]
An operator $M : \ell_2 \to \ell_2$ is said to be *doubly dominant* if the elements $m_{ij}$ of its associated matrix have the following properties.

$$m_{ii} \geq \sum_{j=-\infty, j\neq i}^{\infty} |m_{ij}|, \quad m_{ii} \geq \sum_{j=-\infty, j\neq i}^{\infty} |m_{ji}| \quad \forall i$$

If, in addition, it also holds that

$$m_{ij} \leq 0,\ \forall i \neq j$$

then $M$ said to be *doubly hyperdominant*. For an operator $M : \mathcal{L}_2 \to \mathcal{L}_2$, these notions are defined in terms of its kernel in an analogous manner with integrals suitably replacing sums. □

*Definition 5:* [monotone nonlinearity]
The class $\mathcal{N}_M$ of MIMO *monotone* nonlinearities consists of all memoryless mappings $N : \mathbb{R}^p \to \mathbb{R}^p$ such that $N(x)$ is the gradient of some convex function $P : \mathbb{R}^p \to \mathbb{R}$ and $\exists C \in \mathbb{R}^+ \ni \|N(x)\| \leq C\|x\|$. $\mathcal{N} \doteq \{N \in \mathcal{N}_M | N(0) = 0\}$, $\mathcal{N}_{odd} \doteq \{N \in \mathcal{N} | N(x) = -N(-x)\ \forall x\}$. □

*Definition 6:* [repeated SISO monotone nonlinearity]
The class of *repeated SISO monotone* nonlinearities is the subclass $\mathcal{N}^{RS}$ of $\mathcal{N}$ with element $N \in \mathcal{N}^{RS}$ of the form

$$N(\zeta) \doteq [\phi(\zeta_1)\ \phi(\zeta_2)\ \ldots \phi(\zeta_p)]^T \quad \forall \zeta \in \mathbb{R}^p \tag{1}$$

where $\phi \in \mathcal{N},\ \phi$ SISO. A shorthand notation for (1) is $N = \mathrm{diag}(\phi)$. The class $\mathcal{N}^{RS}_{odd}$ is defined by replacing $\mathcal{N}$ in the definition of $\mathcal{N}^{RS}$ by $\mathcal{N}_{odd}$. □

*Definition 7:* [multipliers]
$\mathcal{M}^{RS}_{odd}$ denotes the class of MIMO convolution operators, either continuous or discrete, such that the impulse response of an $M \in \mathcal{M}^{RS}_{odd}$ is of the form

$$m = g\ \delta - h \tag{2}$$

where $g, h(\cdot) \in \mathbb{R}^{p\times p}$ satisfy

$$g_{ii} \geq \sum_{i=1, i\neq j}^{n} |g_{ij}| + \sum_{i=1}^{n} \|h_{ij}\|_1 \qquad \forall i = 1, 2, \ldots, n \tag{3}$$

$$g_{ii} \geq \sum_{j=1, j\neq i}^{n} |g_{ij}| + \sum_{j=1}^{n} \|h_{ij}\|_1 \qquad \forall i = 1, 2, \ldots, n. \tag{4}$$

The subclass $\mathcal{M}^{RS}$ is obtained by further stipulating

$$g_{ij} \leq 0 \ \forall i \neq j, \quad h_{ij}(\cdot) \geq 0 \ \forall i, j. \tag{5}$$

Under the restriction

$$g,\ h \text{ are Hermitian matrices,} \tag{6}$$

the subclass $\mathcal{M}^D$ ($\mathcal{M}^D_{odd}$) is derived from $\mathcal{M}^{RS}$ ($\mathcal{M}^{RS}_{odd}$). □

*Remark 1:* D'Amato et al [1] showed that $\mathcal{M}^D$ ($\mathcal{M}^D_{odd}$) preserves positivity of $\mathcal{N}^{RS}$ ($\mathcal{N}^{RS}_{odd}$). ■

## III. Problem Formulation

*Problem 1:* Find the largest class of bounded linear operators and the largest class of bounded convolution operators that preserve positivity of *every* nonlinearity in $\mathcal{N}^{RS}$ ($\mathcal{N}^{RS}_{odd}$). □

## IV. Background Results

Paraphrased for notational ease, the main result of D'Amato et al, viz. [1, Theorem 1], is as follows.

*Lemma 1:* `[1, D'Amato et al]`
$\mathcal{M}^D$ ($\mathcal{M}^D_{odd}$) is positivity preserving for $\mathcal{N}^{RS}$ ($\mathcal{N}^{RS}_{odd}$). □
The above result is stated a sufficiency condition and it is not made clear if there exists a larger class of bounded convolution operators that preserves positivity of $\mathcal{N}^{RS}$ ($\mathcal{N}^{RS}_{odd}$). In this regard, it is worthwhile to note the following interesting SISO case result of Willems (see [9, Theorem 3.11, pp. 63]). For easy reading, its statement is slightly modified.

*Lemma 2:* `[9, Willems]`
Let $M : \ell_2 \to \ell_2$ be a bounded linear operator. Then, $\langle x, My\rangle$ is nonnegative for all similarly ordered unbiased (similarly ordered symmetric unbiased) sequences $\{x\}, \{y\} \in \ell_2$ if and only if $M$ is doubly hyperdominant (doubly dominant). □

## V. Main Result

*Theorem 1:* [Solution to Problem 1]
A bounded linear operator $M$ mapping $\ell_2^p$ into $\ell_2^p$ [or $\mathcal{L}_2$ into $\mathcal{L}_2$] preserves positivity of every $N \in \mathcal{N}^{RS}$ ($N \in \mathcal{N}^{RS}_{odd}$) if and only if its associated matrix [kernel] is doubly hyperdominant (doubly dominant). Furthermore, a bounded convolution operator $M$ mapping $\mathcal{L}_2$ into $\mathcal{L}_2$, or mapping $\ell_2^p$ into $\ell_2^p$, preserves positivity of every $N \in \mathcal{N}^{RS}$ ($N \in \mathcal{N}^{RS}_{odd}$) if and only if $M \in \mathcal{M}^{RS}$ ($M \in \mathcal{M}^{RS}_{odd}$). □

*Proof:* We shall prove the result for $\mathcal{N}^{RS}$. The case for $\mathcal{N}^{RS}_{odd}$ follows on similar lines. First, the result will be proved for the discrete time case.

An $N \in \mathcal{N}^{RS}$ can be expressed as $N = \text{diag}(\phi)$ where $\phi \in \mathcal{N}$, $\phi$ SISO. Given sequences $\{x_i\}$ of real valued scalars, define $y_i = \phi(x_i)$ $i = 1, 2, \ldots, p$. Note that the sequences $\{x_i\}$ and $\{y_i\}$ are similarly ordered and unbiased for all $i$ since $N \in \mathcal{N}$. Define $\widetilde{x}(k) \doteq [x_1(k)\ x_2(k)\ \ldots\ x_p(k)]^T$, $\widetilde{y}(k) \doteq [y_1(k)\ y_2(k)\ \ldots\ y_p(k)]^T$ for all $k \in \mathbb{Z}$. Note that the sequences $\{\widetilde{x}\}$ and $\{\widetilde{y}\}$ are similarly ordered and unbiased. Observe that $\langle x, My\rangle = \left\langle \widetilde{x}, \widetilde{M}\widetilde{y}\right\rangle$ where the sequences $\{x\}$ and $\{y\}$ are defined by

$$x(k) \doteq [x_1(k)\ x_2(k)\ \ldots\ x_p(k)]^T,\ y(k) \doteq [y_1(k)\ y_2(k)\ \ldots\ y_p(k)]^T \ \ \forall k$$

and $\widetilde{M} : \ell_2 \to \ell_2$ with its associated matrix same as the associated matrix $\widetilde{M}$ of $M$. Since $M$ is bounded, $\widetilde{M}$ is a bounded operator as well. By Lemma 2, $\left\langle \widetilde{x}, \widetilde{M}\widetilde{y} \right\rangle$ is nonnegative if and only if $\widetilde{M}$ is doubly hyperdominant. This is proves the result for bounded linear operators. To prove the result for bounded convolution operators, note that the associated matrix $\widetilde{M}$ of a bounded convolution operator $M$ is block Toeplitz. Since $\widetilde{M}$ is block Toeplitz, the double hyperdominance conditions need only be checked on a block of $p$ columns and on a block of $p$ rows. The conditions are precisely the ones given by (3)-(5).

To prove the result in the continuous time case, note that continuous time signals $x_c, y_c$ can be sampled with sampling interval $\epsilon$ to produce discrete time signals in $\ell_2$, say $x_{d,\epsilon}, y_{d,\epsilon}$ such that $x_{d,\epsilon}(k) = \sqrt{\epsilon}\, x_c(k\epsilon),\; y_{d,\epsilon}(k) = \sqrt{\epsilon}\, y_c(k\epsilon)$. Likewise, given a continuous time linear operator $M_c : \mathcal{L}_2 \to \mathcal{L}_2$,

$$z_c(t) \doteq \int_{-\infty}^{\infty} \overline{m}_c(t,\tau) y(\tau)\, d\tau \;\forall t,$$

may be discretized as $z_{d,\epsilon} \doteq M_{d,\epsilon} y_{d,\epsilon}$ i.e. as

$$z_{d,\epsilon}(k) \doteq \sum_{l=-\infty}^{\infty} \overline{m}_{k,l} y_{d,\epsilon}(l) \quad \forall k$$

where, for all $k$ and $l$,

$$\overline{m}_{kl} \doteq \frac{1}{\epsilon} \int_{t\in((k-1)\epsilon,k\epsilon]} \int_{\tau\in((l-1)\epsilon,l\epsilon]} \overline{m}_c(t,\tau)\, d\tau\, dt.$$

With this discretization, the continuous time inner-product $\langle x_c, M_c y_c \rangle$ is then recoverable as the limit

$$\langle x_c, M_c y_c \rangle = \lim_{\epsilon\to 0} \langle x_{d,\epsilon}, M_{d,\epsilon} y_{d,\epsilon} \rangle .$$

Taking the limit as $\epsilon \to 0$, the continuous time case proof then follows using the discrete time case arguments. QED.

*Remark 2:* By Theorem 1, $\mathcal{M}^{RS}_{odd}$ preserves positivity of the identity matrix. Thus, every element $M$ of $\mathcal{M}^{RS}_{odd}$ has the property that $\text{herm}(\widehat{m}(\cdot)) \geq 0$. ∎

*Remark 3:* From Theorem 1, it follows that every convolution operator that preserves positivity of $\mathcal{N}^{RS}$ ($\mathcal{N}^{RS}_{odd}$) and, *in addition*, has a Hermitian frequency response matrix is an element of $\mathcal{M}^{D}$ ($\mathcal{M}^{D}_{odd}$). However, $\mathcal{M}^{D}$ ($\mathcal{M}^{D}_{odd}$) is strictly a subset of $\mathcal{M}^{RS}$ ($\mathcal{M}^{RS}_{odd}$) because the condition (6) stipulated by D'Amato et al [1] is unnecessary for the positivity preservation. ∎

## VI. Discussion

By letting go of the unnecessary condition (6) stipulated by D'Amato et al [1], we have obtained a larger — indeed, the largest possible — class $\mathcal{M}^{RS}$ ($\mathcal{M}^{RS}_{odd}$) of multipliers that preserve positivity of $\mathcal{N}^{RS}$ ($\mathcal{N}^{RS}_{odd}$). However, an incremental improvement can be claimed only if it can be demonstrated that use of this larger class of multipliers leads to a further reduction in conservativeness of IQC stability analysis. The following example demonstrates that the conservatism is indeed further reduced.

*Example 1:* Consider the feedback system $\mathcal{S}$ (see Fig. 1) in which $N \in \mathcal{N}^{RS}$ and $H$ is the trivial stable memoryless linear operator having constant frequency response

$$\widehat{h}(\omega) = \begin{bmatrix} 0.5 & -1 \\ 2 & -0.6 \end{bmatrix} \quad \forall\omega.$$

The objective is to determine if the system is stable. Stability is established if (cf. [6, Theorem 2], [8, Theorem 1]) there exists a positivity preserving multiplier $M$ such that the operator $MH$ is positive, i.e.

$$\text{herm}\left(\widehat{m}(\omega)\widehat{h}(\omega)\right) > 0 \quad \forall\omega. \tag{7}$$

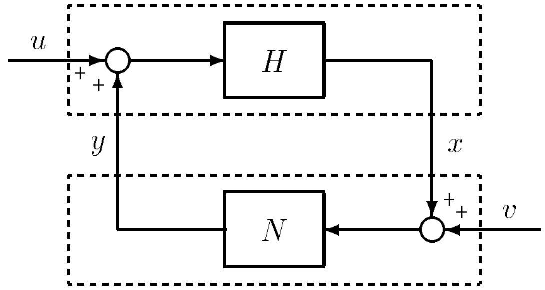


Fig. 1. The feedback system $\mathcal{S}$. $H$ is stable, causal and linear time invariant; $N \in \mathcal{N}_{odd}^{RS}$.

It may be checked that $-\widehat{h}(\omega) \in \mathbb{R}^{2\times 2}$ is not Hurwitz so that, by Lyapunov Criterion (see, e.g., [12, Theorem 2.6-1]), herm $\left(P\widehat{h}(\omega)\right) \not> 0$ for any $P = P^T > 0$. In view of the condition (7), it then follows that stability *cannot* be determined using $M \in \mathcal{M}^D$ since (see Remark 2) $\widehat{m}(\omega) = \widehat{m}(\omega)^T \geq 0$ for every $M \in \mathcal{M}^D$. On the other hand, choosing the *asymmetric* multiplier

$$M = \begin{bmatrix} 1 & 0 \\ -0.9 & 1 \end{bmatrix}, \tag{8}$$

which may be verified to be in $\mathcal{M}^{RS}$ (so that $M \in \mathcal{M}_{odd}^{RS}$ as well), the condition (7) is satisfied so that stability of the system *is* established using the class $\mathcal{M}^{RS}$ derived in this note. Even if the nonlinearity $N$ were odd, using similar arguments it follows that stability cannot be determined using $M \in \mathcal{M}_{odd}^D$ whereas it can be determined using the multiplier given by (8). This demonstrates an example in which $\mathcal{M}^{RS}$ ($\mathcal{M}_{odd}^{RS}$) leads to a strictly less conservative stability analysis than $\mathcal{M}^D$ ($\mathcal{M}_{odd}^D$). ■

*Remark 4:* The above example shows that our main result incrementally improves upon the main result in [1]. Its utility can be seen via the same engineering application considered by D'Amato et al [1, *Automatica* version]. ■

## VII. Conclusion

We have derived the largest possible class of bounded MIMO linear operators that preserve positivity of repeated monotone nonlinearities. Also derived are the largest possible classes $\mathcal{M}^{RS}$ and $\mathcal{M}_{odd}^{RS}$ of bounded MIMO convolution operators that preserve positivity of repeated monotone and repeated odd monotone nonlinearities, respectively. It follows that (see Remark 2) *modulo the restriction* that multipliers have a Hermitian frequency response for all frequencies, D'Amato et al [1] have actually derived the largest possible classes of multipliers that preserve positivity of such nonlinearities. The less restrictive nature of our multipliers produces less conservative stability results. We have demonstrated this conservativeness reduction via an example.

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