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Incremental Positivity Preservation Properties of Stability Multipliers

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Abstract

It is proved that stability multipliers such as Zames-Falb multipliers, Popov multipliers and RL/RC multipliers, known to preserve positivity of monotone memoryless nonlinearities, do not, in general, preserve their incremental positivity.

1 Introduction

In stability analysis, a given system $\mathcal{S}$ is often decomposed into two interconnected subsystems — a linear time invariant subsystem $\mathcal{S}_1$ in the feedforward path and an otherwise subsystem $\mathcal{S}_2$ in the feedback path. Stability of $\mathcal{S}$ is then deduced if there exists a quadratic functional that separates the graph of $\mathcal{S}_1$ from the graph of $\mathcal{S}_2$. Certain classes of convolution operators, also called *stability multipliers*, specify such functionals. The structure of such a multiplier is governed by the structure of $\mathcal{S}_2$ and the multiplier is referred to as a stability multiplier *for* $\mathcal{S}_2$. Prominent among such multipliers for *monotone* nonlinearities are Zames-Falb multipliers [1] and their limiting cases such as Popov multipliers [2] and RL/RC multipliers [3].

According to a definition given by Zames in 1966 [4], in order for an input-output system to be considered *stable* it must have two properties:

1) it must be *bounded*, i.e. bounded inputs produce bounded outputs, and
2) it must be *continuous*, i.e. it is not critically sensitive to small changes in inputs.

Despite this, the now famous Zames-Falb multiplier paper [1] published just two years later in 1968 described stability conditions solely in terms of boundedness (see [1, Theorem 1 and 2]); it neither addressed nor even mentioned the issue of continuity. Similarly, Popov's celebrated 1961 work [2] concerns asymptotic stability of the null solution and not continuity of a system, as does the well known 1968 work of Cho and Narendra [3]. Whether a connection exists between continuity of a system and the conditions laid down by these multipliers for its stability has been a question scarcely investigated.

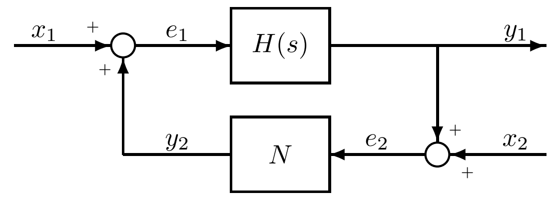

Figure 1: The feedback system Ξ considered by Zames. The system is said to be stable if the mapping from the input signal $x = \text{col}(x_1, x_2)$ to the output $y = \text{col}(e_1, e_2)$ is both bounded and continuous.

Stability multipliers are used to reduce conservatism in stability analysis of the system shown in Fig. 1. The key relevant property of a stability multiplier M is that it is *positivity preserving* for monotone nonlinearities N in the sense that the operator MN is positive. Loosely speaking, if the operator M^*H is strongly positive for some such M then boundedness of the system is established. To establish continuity, Zames [4, Theorem 3] strengthened the requirement of positivity to that of *incremental positivity*, showing that if H and N are incrementally positive, then the closed-loop system is stable (i.e., both continuous and bounded).

Now, every monotone nonlinearity is incrementally positive so, should the stability multipliers turn out to be *incremental* positivity preserving for monotone nonlinearities, then it would be relatively straightforward to establish continuity of the system shown in Fig. 1. For this reason, whether these multipliers are incremental positivity preserving for monotone nonlinearities is a question of considerable interest. Our main results, viz. Theorem 1 and Corollary 1, establish that they are are not.

2 Background and Notation

The notation used is summarized in Table 1. Capital letter symbols, e.g. F and G, denote operators whereas

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Table 1: Notation

Symbol	Meaning
$(\mathbb{R}^+)$ $\mathbb{R}$	Set of all (nonnegative) real numbers.
$\mathbb{Z}$	Set of all integers.
$\langle x, y \rangle$	$= \int_{-\infty}^{\infty} y^T(t)x(t)dt.$
$\Vert x \Vert$	$= \sqrt{\langle x, x \rangle}.$
$\mathcal{L}_2$	Space of possibly vector valued signals x for which $\Vert x \Vert < \infty$.
$\Vert z \Vert_{\mathcal{L}_1}$	$= \int_{-\infty}^{\infty} \vert z(t) \vert \, dt.$
$x(t) * y(t)$	$= \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau,$ (convolution).
x^*	$x^*(t) = x^T(-t)$ if $x(t) \in \mathbb{R}^n$.
$\widehat{x}$	$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$ (Fourier transform).
$r_{xy}(t)$	$= x * y^* = \int_{-\infty}^{\infty} x(t+\tau)y^T(\tau)d\tau,$ (correlation function).
D_τ	$(D_\tau x)(t) \doteq x(t-\tau) \quad \forall t \ (\tau \in \mathbb{R}),$ (time-delay operator).
$\delta(t)$	$= \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{else.} \end{cases}$ (delta function).
$P'(x)$	Gradient of a real-valued function $P(x)$.
$O(\epsilon)$	Of the order of ϵ.
MIMO	multi-input multi-output.
SISO	single-input-single-output.

small letters, e.g. x and y, denote vectors or matrices of real signals. $\cup$ is the set union operator. We say that a differentiable nonlinear function $N : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if for some convex differentiable function $P : \mathbb{R}^n \to \mathbb{R}$ it holds that N is the gradient of P; i.e.,

$$N(x) = P'(x) \ \forall x \in \mathbb{R}^n. \tag{1}$$

For a differentiable N, the monotonicity property is equivalent to N having a positive semidefinite, Hermitian Jacobian matrix (see [5]); i.e. it is equivalent to the property

$$\frac{\partial}{\partial x}N(x) = \left(\frac{\partial}{\partial x}N(x)\right)^T \geq 0 \quad \forall x \in \mathbb{R}^n. \tag{2}$$

If $N(-x) = -N(x)$ $\forall x \in \mathbb{R}^n$ then we say that N is *odd*.

Definition 1 *$\mathcal{N}$ (or $\mathcal{N}_{odd}$) denotes the class of (odd) monotone nonlinearities N for which $N(0) = 0$.* □

Definition 2 *An operator $F : \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ is termed* positive *if $\langle x, Fx \rangle \geq 0$ $\forall x \in \mathcal{L}_2$. If*

$$\langle x - y, Fx - Fy \rangle \ \geq 0 \quad \forall x, y \in \mathcal{L}_2, \tag{3}$$

then it is termed incrementally positive. □

Remark 1 For differentiable SISO nonlinearities, it is well known that monotonicity and incremental positivity are equivalent. In the MIMO case, it remains true that every monotone nonlinearity N is incrementally positive, but the converse is not true [5]. ■

Definition 3 [stability multipliers]
$\mathcal{M}_P$ denotes the class of convolution operators $M : x \mapsto m * x$ with

$$\widehat{m}(j\omega) \doteq 1 \ + \ q \ j\omega \quad \forall \omega \quad \text{where } q \geq 0.$$

The elements of $\mathcal{M}_P$ are called Popov multipliers *[2].*
$\mathcal{M}$ (or $\mathcal{M}_{odd}$) denotes the class of convolution operators $M : x \mapsto m * x$ with $\widehat{m}(j\omega) \doteq m_0 - \widehat{z}(j\omega)$ $\forall \omega$ where $m_0 - \Vert z \Vert_{\mathcal{L}_1} > 0$ and $z(t) \in \mathbb{R}^+$ $\forall t$ (or, respectively, $z(t) \in \mathbb{R}$ $\forall t$). The elements of $\mathcal{M}$ and $\mathcal{M}_{odd}$ are called Zames-Falb multipliers *[1].*
$\mathcal{M}_{RL}$ denotes the class of convolution operators $M : x \mapsto m * x$ with Fourier transform

$$\widehat{m}(j\omega) = \Pi_{n=0}^{N} \frac{j\omega + \alpha_n}{j\omega + \beta_n}$$

where $0 < \alpha_0 < \beta_0 < \alpha_1 < \ldots < \alpha_N < \beta_N$. $\mathcal{M}_{RC}$ denotes the class of convolution operators $M : x \mapsto m * x$ such that $M^{-1} \in \mathcal{M}_{RL}$. Elements of $\mathcal{M}_{RL}$ and $\mathcal{M}_{RC}$ are called RL *and* RC multipliers*, respectively [3] and $\mathcal{M}_M \doteq \mathcal{M}_P \cup \mathcal{M} \cup \mathcal{M}_{RL} \cup \mathcal{M}_{RC}$.* □

The key property of these stability multipliers is that they are *positivity preserving* for monotone nonlinearities in that (see [1, 5])

$$MN \text{ is positive } \forall N \in \mathcal{N}, M \in \mathcal{M}_M \tag{4}$$

$$MN \text{ is positive } \forall N \in \mathcal{N}_{odd}, M \in \mathcal{M}_{odd}. \tag{5}$$

From a stability and robustness analysis perspective, the importance of the these multiplier positivity relations (4)-(5) is that they determine a class of IQC's for monotone nonlinearities (see [6] for IQC theory and applications). The resulting IQC stability characterization is the sharpest available for monotone nonlinearities; and it may well be the sharpest possible set of IQC's obtainable using convolution operators (cf. Willems [7, Thm. 3.13]). Generalizations of the Zames-Falb multiplier theory have also led to improved IQC's for the special subclass of $\mathcal{N}$ in which $N : \mathbb{R}^n \to \mathbb{R}^n$ consists of n repeated SISO nonlinearities [8]. Moreover, in recent years there has been a development of software packages, such as [9], using which the stability conditions stipulated by $\mathcal{M}$ can be tested (see, e.g., the results in [10], [11] and [12]). Subsequently, $\mathcal{M}$ has received a considerable attention (see [5], [8], [13] and references therein). Yet, whether $\mathcal{M}$ (or $\mathcal{M}_{odd}$) is *incremental positivity* preserving for $N \in \mathcal{N}$ (or, respectively, $N \in \mathcal{N}_{odd}$), has until now remained an open question. The question is, "Do (4) or (5) still hold when the term *positivity* is replaced by *incremental positivity*?"

Our main results, viz. Theorem 1 and Corollary 1, establish that they are are not.

3 Main Results

We shall first establish a key preliminary result.

Lemma 1 *Suppose* $N \in \mathcal{N}, x, \bar{x} \in \mathcal{L}_2, \tau \in \mathbb{R}, \tau \neq 0$. *Let* $\widetilde{x} \doteq x - \bar{x}$ *and* $\widetilde{y} \doteq N(x) - N(\bar{x})$. *Then,*

$$\text{trace}[r_{\widetilde{y}\widetilde{x}}(0)] \geq 0 \quad \forall x, \bar{x} \in \mathcal{L}_2. \tag{6}$$

Furthermore,

$$\text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)] \geq 0 \quad \forall x, \bar{x} \in \mathcal{L}_2 \ \forall \tau \in \mathbb{R} \tag{7}$$

holds if and only if N *is linear.* □

Proof: The inequality (6) follows from (3) and the fact that an $N \in \mathcal{N}$ has a finite incremental gain. It remains to prove that (7) holds if and only if N is linear.

(**if**) Suppose that N is linear. Then for all $x, \bar{x} \in \mathcal{L}_2$,

$$\begin{aligned} \text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)] &= \langle M\left(N(x) - N(\bar{x})\right), x - \bar{x}\rangle \\ &= \langle MN(\widetilde{x}), \widetilde{x}\rangle \\ &\geq 0 \end{aligned}$$

where the last inequality follows since MN is positive — see (4)-(5).
(**only if**) Suppose $N \in \mathcal{N}$, N not linear. It needs to be shown that for every such N, there exist $x \in \mathcal{L}_2$ and $\bar{x} \in \mathcal{L}_2$ for which (7) does not hold. Let us first consider the case in which N is SISO.

Assume without loss of generality that N is differentiable. If $N \in \mathcal{N}$ is not linear, then there exist two points $\zeta_1, \zeta_2 \in \mathbb{R}$, $\zeta_1 \neq \zeta_2$ such that

$$\beta \doteq N'(x)\Big|_{x=\zeta_1} > \alpha \doteq N'(x)\Big|_{x=\zeta_2} \tag{8}$$

where $\alpha \geq 0$ since $N \in \mathcal{N}$. Existence of signals $x, \bar{x}$ for which (7) fails to hold needs to be demonstrated. For that purpose, choose the test signals $x, \bar{x}$ to be square-wave signal defined as follows.

$$\bar{x}(t) = \begin{cases} \zeta_1 & \text{if } t \in \mathcal{T}_1 \\ \zeta_2 & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} ; \; x(t) = \begin{cases} \bar{x}(t) + \epsilon\mu & \text{if } t \in \mathcal{T}_1 \\ \bar{x}(t) + \epsilon & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} \tag{9}$$

where, with $k \in \mathbb{Z}$, $\mathcal{T}_1 \doteq \bigcup_{k=0}^{\ell-1} [2k\delta, (2k+1)\delta)$, and $\mathcal{T}_2 \doteq \bigcup_{k=0}^{\ell-1} [(2k+1)\delta, (2k+2)\delta)$, $\mu = \frac{\alpha+\beta}{2\beta}$; the choices of ϵ and $\ell > 0$ will be explained shortly, $\nu \doteq \delta/\tau > 1, \nu \in \mathcal{Z}$.

For $x, \bar{x}$ as chosen above, $\widetilde{x}$ and $\widetilde{y}$ are given by

$$\widetilde{x}(t) = \begin{cases} \epsilon\mu & \text{if } t \in \mathcal{T}_1 \\ \epsilon & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} ; \; \widetilde{y}(t) = \begin{cases} \beta\epsilon\mu + O(\epsilon^2) & \text{if } t \in \mathcal{T}_1 \\ \alpha\epsilon + O(\epsilon^2) & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} .$$

Let $a, b, v, u \in \mathbb{R}^\nu$ be defined as $b \doteq [\beta \ \beta \ \ldots \ \beta]^T$, $a \doteq [\alpha \ \alpha \ \ldots \ \alpha]^T$, $v \doteq [\mu \ \mu \ \ldots \ \mu]^T$, $u \doteq [1 \ 1 \ \ldots \ 1]^T$. Let $Q \in \mathbb{R}^{2\ell\nu \times 2\ell\nu}$ have all its entries zero except for the diagonal which is given by $[b \ a \ \ldots b \ a]$ and the first sub-diagonal which is given by $[-b \ -a \ \ldots \ -b \ -a \ -b]$. Let $w \doteq [v \ u \ \ldots \ v \ u]^T$, $w \in \mathbb{R}^{2\ell\nu}$. It may then be verified that

$$\begin{aligned} \text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)] &= \langle \widetilde{y}, (I - D_\tau)\widetilde{x}\rangle \\ &= \epsilon^2\tau \ w^T Q w \ + \ O(\epsilon^3) \\ &= -(\ell-1)\epsilon^2\tau[O(1/\ell) + \frac{(\beta-\alpha)^2}{4\beta} + O(\epsilon)]. \end{aligned} \tag{10}$$

Choosing ℓ sufficiently large and ϵ sufficiently small, (10) implies $\text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)] < 0$. This proves the 'only if' part of (7) for the SISO case.

For a MIMO N, the 'only if' part of (7) can be seen to hold as follows. Again, assume without loss of generality that N is differentiable; i.e., the Jacobian matrix $N'(x) = \frac{\partial}{\partial x}N(x)$ exists. Since N is not linear, there exist vectors $\zeta_1, \zeta_2 \in \mathbb{R}^n$ such that the Jacobian matrix of N evaluated at ζ_1 differs from the Jacobian matrix of N evaluated at ζ_2 in at least one entry, say in the (i,j)-th entry. Then, the SISO case arguments, modified by applying the perturbations $\epsilon\mu$ and μ to only the j-th component of $\bar{x}$, may be seen to lead to the desired result. QED.

Lemma 2 *Suppose* $N \in \mathcal{N}$ *and* $x, \bar{x} \in \mathcal{L}_2$. *Let* $\widetilde{x} \doteq x - \bar{x}$, $\widetilde{y} \doteq N(x) - N(\bar{x})$. *Let* $\widetilde{v}(t) \doteq \frac{d}{dt}\widetilde{y}(t) \ \forall t$. *Then,*

$$\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] \geq 0 \quad \forall x, \bar{x} \in \mathcal{L}_2 \tag{11}$$

holds if and only if N *is linear.* □

Proof:
(if) Suppose that N is linear. First note that

$$\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] = \lim_{\tau \to 0} \frac{\text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)]}{\tau} \quad \forall x, \bar{x} \in \mathcal{L}_2 \ \forall \tau \in \mathbb{R}. \tag{12}$$

The result then follows using (7).

(only if) Suppose that N is not linear, N SISO. Choose $x, \bar{x} \in \mathcal{L}_2$ as given by (9). Then (10) and (12) imply that

$$\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] = -(\ell-1)\epsilon^2[O(1/\ell) + \frac{(\beta-\alpha)^2}{4\beta} + O(\epsilon)]. \tag{13}$$

Taking ϵ small enough and ℓ large enough, the result follows. The 'only if' part of (11) for a MIMO N follows as outlined on the lines of the 'only if' part of (7) for a MIMO N. QED.

Theorem 1 *Let $N \in \mathcal{N}$. Then, MN is incrementally positive for every $M \in \mathcal{M}_P$ if and only if N is linear.* □

Proof: Let $x, \overline{x} \in \mathcal{L}_2$ and write $y \doteq N(x), \overline{y} \doteq N(\overline{x}), \tilde{x} \doteq x - \overline{x}$ with the corresponding $\tilde{y} \doteq y - \overline{y}$.

(if) Suppose that N is linear. In this case, $N(\tilde{x}) = N(x) - N(\overline{x})$. By (4)-(5), we have

$$\begin{aligned} &\langle M\left(N(x) - N(\overline{x})\right), x - \overline{x}\rangle \\ &\quad = \langle MN(\widetilde{x}), \widetilde{x}\rangle \geq 0 \quad \forall x, \overline{x} \in \mathcal{L}_2; \end{aligned} \tag{14}$$

that is, $M \in \mathcal{M}$ ($M \in \mathcal{M}_{odd}$) is incrementally positivity preserving for all *linear* $N \in \mathcal{N}$ ($N \in \mathcal{N}_{odd}$).

(only if) Now suppose that N is not linear. Consider an $M_* \in \mathcal{M}_P$, its parameter q to be described shortly. Note that

$$\begin{aligned} \langle M^*\left(N(x) - N(\overline{x})\right), x - \overline{x}\rangle &= \langle \widetilde{y}, \widetilde{x}\rangle + q\,\langle \widetilde{v}, \widetilde{x}\rangle \\ &= \operatorname{trace}[r_{\widetilde{y}\widetilde{x}}(0)] + q \operatorname{trace}[r_{\widetilde{v}\widetilde{x}}(0)] \end{aligned} \tag{15}$$

By Lemma 2, there exists $\{x_* \in \mathcal{L}_2, \overline{x}_* \in \mathcal{L}_2, t_* \in \mathbb{R}\}$ such that

$$\gamma_1 \doteq \operatorname{trace}[r_{\widetilde{y}_*\widetilde{x}_*}(0)] \geq 0, \tag{16}$$

$$\gamma_2 \doteq \operatorname{trace}[r_{\widetilde{v}_*\widetilde{x}_*}(0)] < 0 \tag{17}$$

where $\widetilde{x}_* \doteq x_* - \overline{x}_*$ with the corresponding $\widetilde{y}_* \doteq N(x_*) - N(\overline{x}_*)$ and $\widetilde{v}_* \doteq v_* - \overline{v}_*$. Choose the parameter q of the candidate multiplier M_* as

$$q > \frac{\gamma_1}{-\gamma_2}. \tag{18}$$

Note that $q > 0$ so that $M_* \in \mathcal{M}_P$. Also, $\gamma_1 \doteq \operatorname{trace}[r_{\widetilde{y}_*\widetilde{x}_*}(0)]$ is a bounded nonnegative quantity (see (6)). Whence, it follows using (15)–(18) that

$$\begin{aligned} \langle M_*\left(N_*(x_*) - N_*(\overline{x}_*)\right), x_* - \overline{x}_*\rangle &= \gamma_1 + q\gamma_2 \\ &< 0. \quad \text{QED.} \end{aligned}$$

Corollary 1 *Let $N \in \mathcal{N}$ (or, respectively, $N \in \mathcal{N}_{odd}$). Then, MN is incrementally positive for every $M \in \mathcal{M}$ ($M \in \mathcal{M}_{odd}$) if and only if N is linear. Furthermore, every M in $\mathcal{M}_{RL}$ (and, alternatively, $M \in \mathcal{M}_{RC}$) is incremental positivity preserving for an $N \in \mathcal{N}$ if and only if N is linear.* □

Proof: Let $x, \overline{x} \in \mathcal{L}_2$ and write $y \doteq N(x), \overline{y} \doteq N(\overline{x}), \tilde{x} \doteq x - \overline{x}$ with the corresponding $\tilde{y} \doteq y - \overline{y}$.

(if) Follows on the lines of the proof of the (if) part of Theorem 1.

(only if) Now suppose that N is not linear. Consider the multiplier $M : x \mapsto m * x$ with

$$\widehat{m}(j\omega) = \frac{1 + qj\omega}{\epsilon j\omega + 1} \quad \forall \omega \in \mathbb{R}; \quad q, \epsilon > 0. \tag{19}$$

By continuity of inner product (see [14, Chapter 3, pp. 138]),

$$\begin{aligned} &\lim_{\epsilon \to 0} \langle M^*\left(N(x) - N(\overline{x})\right), x - \overline{x}\rangle \\ &\quad = \langle \widetilde{y}, \widetilde{x}\rangle + q\,\langle \widetilde{v}, \widetilde{x}\rangle \\ &\quad = \operatorname{trace}[r_{\widetilde{y}\widetilde{x}}(0)] + q \operatorname{trace}[r_{\widetilde{v}\widetilde{x}}(0)]. \end{aligned} \tag{20}$$

Choosing the signals and the parameter q as described in (16)-(18), it follows that for vanishingly small ϵ, the RL multiplier M specified by (19) and (18) fails to preserve incremental positivity of the nonlinearity N. It then follows that its inverse, which is an RC multiplier, fails to preserve incremental positivity of the inverse of N, which is also an element of $\mathcal{N}$. Since $\mathcal{M}_{RC} \subset \mathcal{M} \subset \mathcal{M}_{odd}$ [7], the result also holds for Zames-Falb multipliers $\mathcal{M}$ and $\mathcal{M}_{odd}$. QED.

4 Conclusion

The question of whether Popov, Zames-Falb and related dynamical stability multipliers may fail to preserve the incremental positivity of monotone nonlinearities has been examined and answered, both for SISO and MIMO nonlinearities. They may fail. Theorem 1 and Corollary 1 together establish that for every monotone nonlinearity N that is not linear, there exists a multiplier M in each of such multiplier classes such that MN is not incrementally positive. This leads to the conjecture that there may exist a system that satisfies stability conditions laid out by these multipliers and yet exhibits a discontinuous behavior, inconsistent with Zames' [4] definition of input-output stability.

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