Incremental Positivity Non-Preservation by Stability Multipliers

Vishwesh V. Kulkarni and Michael G. Safonov

Abstract

It is proved that stability multipliers such as Zames-Falb multipliers, Popov multipliers and RL/RC multipliers, known to preserve positivity of monotone memoryless nonlinearities, do not, in general, preserve their incremental positivity. Our result implies that stability results based on these multipliers should be interpreted with caution since without incremental positivity the continuity of the closed-loop system's input-output relation may not be assured.

Keywords

nonlinear stability, input-output, Popov multipliers, Zames-Falb, positivity, integral quadratic constraint (IQC).

I. Introduction

In stability analysis, a given system $\mathcal{S}$ is often decomposed into two interconnected subsystems — a linear time invariant subsystem $\mathcal{S}_1$ in the feedforward path and an otherwise subsystem $\mathcal{S}_2$ in the feedback path. Stability of $\mathcal{S}$ is then deduced if there exists a quadratic functional that separates the graph of $\mathcal{S}_1$ from the graph of $\mathcal{S}_2$. Certain classes of convolution operators, also called *stability multipliers*, specify such functionals. The structure of such a multiplier is governed by the structure of $\mathcal{S}_2$ and the multiplier is referred to as a stability multiplier *for* $\mathcal{S}_2$. Prominent among such multipliers for *monotone* nonlinearities are Zames-Falb multipliers [1] and their limiting cases such as Popov multipliers [2] and RL/RC multipliers [3].

According to a definition given by Zames in 1966 [4], in order for an input-output system to be considered *stable* it must have two properties:

1) it must be *bounded*, i.e. bounded inputs produce bounded outputs, and
2) it must be *continuous*, i.e. it is not critically sensitive to small changes in inputs.

Despite this, the now famous Zames-Falb multiplier paper [1] published just two years later in 1968 describes stability conditions solely in terms of boundedness (see [1, Theorem 1 and 2]); it neither addresses nor even mentions the issue of continuity. Similarly, Popov's celebrated 1961 work [2] concerns asymptotic stability of the null solution and not continuity of a system, as does the well known 1968 work of Cho and Narendra [3]. Whether a connection exists between continuity of a system and the conditions laid down by these multipliers for its stability has been a question scarcely investigated.

Stability multipliers are used to reduce conservatism in stability analysis of the system shown in Fig. 1. The key relevant property of a stability multiplier M is that it is *positivity preserving* for monotone nonlinearities N in the sense that the operator MN is positive. Loosely speaking, if the operator M^*H is strongly positive for some such M then boundedness of the system is established. To establish continuity, Zames [4, Theorem 3] strengthened the requirement of positivity to that of *incremental positivity*, showing that if H and N are incrementally positive, then the closed-loop system is stable (i.e., both continuous and bounded).

Now, every monotone nonlinearity is incrementally positive so, should the stability multipliers turn out to be *incremental* positivity preserving for monotone nonlinearities, then it would be relatively straightforward to establish continuity of the system shown in Fig. 1. For this reason, whether these multipliers are incremental positivity preserving for monotone nonlinearities is a question of considerable interest. Our main results, viz. Theorem 1 and Corollary 1, establish that they are not.

The authors are with the Department of Electrical Engineering—Systems, University of Southern California, Los Angeles, CA 90089-2563. Corresponding author: Michael G. Safonov, Fax +1-213-821-1109. Email msafonov@usc.edu
This research was supported in part by AFOSR grants F49620-98-1-0026 and F49620-01-1-0302.

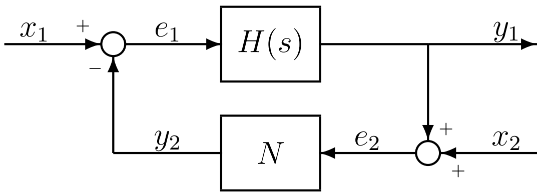

Fig. 1. The feedback system considered by Zames. The system is said to be stable if the mapping from the input signal $x = \mathrm{col}(x_1, x_2)$ to the output $y = \mathrm{col}(e_1, e_2)$ is both bounded and continuous.

TABLE I
NOTATION

Symbol	Meaning	
$(\mathbb{R}^+)$ $\mathbb{R}$	Set of all (nonnegative) real numbers.	
$\mathbb{Z}$	Set of all integers.	
$(\cdot)'$ or $(\cdot)^T$	Transpose of a vector or a matrix $(\cdot)$.	
$\langle x, y \rangle$	$= \displaystyle\int_{-\infty}^{\infty} y^T(t)x(t)dt,$	(inner product).
$\|x\|$	$= \sqrt{\langle x, x \rangle}.$	
$\mathcal{L}_2$	Space of possibly vector valued signals x for which $\|x\| < \infty$.	
$\|z\|_{\mathcal{L}_1}$	$= \int_{-\infty}^{\infty} \lvert z(t) \rvert \, dt.$	
$x(t) * y(t)$	$= \displaystyle\int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau,$	(convolution).
x^*	$x^*(t) = x^T(-t)$ if $x(t) \in \mathbb{R}^n$.	
$\widehat{x}$	$= \displaystyle\int_{-\infty}^{\infty} x(t)e^{-j\omega t} \, dt,$	(Fourier transform).
$r_{xy}(t)$	$= x * y^* = \displaystyle\int_{-\infty}^{\infty} x(t+\tau)y^T(\tau)d\tau,$	(correlation function).
D_τ	$(D_\tau x)(t) \doteq x(t-\tau)$ $\forall t$ $(\tau \in \mathbb{R})$,	(time-delay operator).
$\delta(t)$	$= \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{else.} \end{cases}$	(delta function).
$P'(x)$	Gradient of a real-valued function $P(x)$.	
$O(\epsilon)$	Of the order of ϵ.	
MIMO (SISO)	multi-input multi-output (single-input single-output).	

II. Background and Notation

The notation used is summarized in Table I. Capital letter symbols, e.g. F and G, denote operators whereas small letters, e.g. x and y, denote vectors or matrices of real signals. We say that a differentiable nonlinear function $N : \mathbb{R}^n \to \mathbb{R}^n$ is *monotone* if for some convex differentiable function $P : \mathbb{R}^n \to \mathbb{R}$ it holds that N is the gradient of P; i.e.,

$$N(x) = P'(x) \ \forall x \in \mathbb{R}^n. \tag{1}$$

For a differentiable N, the monotonicity property is equivalent to N having a positive semidefinite, Hermitian Jacobian matrix (see [5]); i.e. it is equivalent to the property

$$\frac{\partial}{\partial x} N(x) = \left(\frac{\partial}{\partial x} N(x) \right)^T \geq 0 \ \ \forall x \in \mathbb{R}^n. \tag{2}$$

If $N(-x) = -N(x)$ $\forall x \in \mathbb{R}^n$ then we say that N is *odd.*

Definition 1: An operator $F : \mathcal{L}_2 \longrightarrow \mathcal{L}_2$ is termed *positive* if $\langle x, y \rangle \geq 0$ $\forall x, y \in \mathcal{L}_2$ such that $y = F(x)$. If

$$\langle x - y, F(x) - F(y) \rangle \ \geq 0 \ \ \forall x, y, F(x), F(y) \in \mathcal{L}_2, \tag{3}$$

then it is termed *incrementally positive.* □

Remark 1: For differentiable SISO nonlinearities, it is well known that monotonicity and incremental positivity are equivalent. In the MIMO case, it remains true that every monotone nonlinearity N is incrementally positive, but the converse is not true [5]. ■

Definition 2: [class $\mathcal{N}$ of monotone nonlinearities]
The class of *norm-bounded positive* (norm-bounded odd and positive) *monotone nonlinearities* $\mathcal{N}$ (or, respectively, $\mathcal{N}_{odd}$) is the class of (odd) monotone nonlinearities N for which

1. $\exists C \in \mathbb{R}$ s.t. $\|N(\zeta)\| \leq C\|\zeta\| \ \ \forall\zeta$ (norm-boundedness); and,
2. $N(0) = 0$ (unbiasedness). □

Remark 2: In the Lemmas and Theorems of this paper, we will refer to nonlinearities in $\mathcal{N}$ as simply *monotone nonlinearities* to facilitate a friendly and quick grasp of the results at the cost of a slight notational abuse. ■

Definition 3: [stability multipliers]
$\mathcal{M}_P$ denotes the class of convolution operators $M : x \mapsto m * x$ with

$$\widehat{m}(j\omega) \doteq (1 \ + \ q \ j\omega)^{\pm 1} \ \ \forall\omega \quad \text{where } q \geq 0.$$

The elements of $\mathcal{M}_P$ are called *Popov multipliers* [2].
$\mathcal{M}$ (or $\mathcal{M}_{odd}$) denotes the class of convolution operators $M : x \mapsto m * x$ with $\widehat{m}(j\omega) \doteq m_0 - \widehat{z}(j\omega) \ \ \forall\omega$ where $m_0 - \|z\|_{\mathcal{L}_1} > 0$ and $z(t) \in \mathbb{R}^+ \ \forall t$ (or, respectively, $z(t) \in \mathbb{R} \ \forall t$). The elements of $\mathcal{M}$ and $\mathcal{M}_{odd}$ are called *Zames-Falb multipliers* [1].
$\mathcal{M}_{RL}$ denotes the class of convolution operators $M : x \mapsto m * x$ with Fourier transform

$$\widehat{m}(j\omega) = \Pi_{n=0}^{N} \frac{j\omega + \alpha_n}{j\omega + \beta_n}$$

where $0 < \alpha_0 < \beta_0 < \alpha_1 < \ldots < \alpha_N < \beta_N$. $\mathcal{M}_{RC}$ denotes the class of convolution operators $M : x \mapsto m * x$ such that $M^{-1} \in \mathcal{M}_{RL}$. Elements of $\mathcal{M}_{RL}$ and $\mathcal{M}_{RC}$ are called *RL* and *RC multipliers*, respectively [3]. Multipliers in $\mathcal{M}_{RL}$, $\mathcal{M}_{RC}$, $\mathcal{M}_P$ and $\mathcal{M}$ are referred to as *stability multipliers.* □

Remark 3: The class of Zames-Falb multipliers includes the RL and RC multipliers. Since

$$1 + qj\omega = \lim_{\epsilon\to 0} \frac{1 + qj\omega}{1 + \epsilon j\omega}, \quad \frac{1}{1 + qj\omega} = \lim_{\epsilon\to 0} \frac{1 + \epsilon j\omega}{1 + qj\omega},$$

Popov multipliers are the limiting case of Zames-Falb multipliers. ■

The key property of these stability multipliers is that they are *positivity preserving* for monotone nonlinearities in that (see [1], [5])

$$MN \text{ is positive } \forall N \in \mathcal{N}, M \text{ a stability multiplier} \tag{4}$$

$$MN \text{ is positive } \forall N \in \mathcal{N}_{odd}, M \in \mathcal{M}_{odd}. \tag{5}$$

From a stability and robustness analysis perspective, the importance of the these multiplier positivity relations (4)-(5) is that they determine a class of IQC's for monotone nonlinearities (see [6] for IQC theory and applications). The resulting IQC stability characterization is the sharpest available for monotone nonlinearities; and it may well be the sharpest possible set of IQC's obtainable using convolution operators (cf. Willems [7, Thm. 3.13]). Generalizations of the Zames-Falb multiplier theory have also led to improved IQC's for the special subclass of $\mathcal{N}$ in which $N : \mathbb{R}^n \to \mathbb{R}^n$ consists of n repeated SISO nonlinearities [8]. Moreover, in recent years there has been a development of software packages, such as [9], using which the stability conditions stipulated by $\mathcal{M}$ can be tested (see, e.g., the results in [10], [11] and [12]). Subsequently, $\mathcal{M}$ has received a considerable attention (see [5], [8], [13] and references therein). Yet, whether $\mathcal{M}$ (or $\mathcal{M}_{odd}$) is *incremental positivity* preserving for $N \in \mathcal{N}$ (or, respectively, $N \in \mathcal{N}_{odd}$), has until now remained an open question. The question is,

"Do (4) or (5) still hold when the term *positivity* is replaced by *incremental positivity*?"

Our main results, viz. Theorem 1 and Corollary 1, establish that they are not.

III. Main Results and Discussion

Theorem 1: Popov multipliers preserve incremental positivity of a monotone nonlinearity if and only if the nonlinearity is linear. □

Proof: See Appendix. QED.

Corollary 1: Zames-Falb multipliers preserve incremental positivity of a monotone nonlinearity if and only if the nonlinearity is linear. Likewise with RL and RC multipliers. □

Proof: See Appendix. QED.

Theorem 1 and Corollary 1 together prove that in general stability multipliers do not preserve incremental positivity of monotone nonlinearities unless the nonlinearities are actually linearities. This casts ambiguity over their reliability concerning continuity determination for systems of the form given in Figure 1. To the best of our knowledge, the issue of continuity determination has not received a rigorous attention thus far although it ought to be mentioned that some of the past work does address special cases of this problem without alluding to them. For example, the proofs in [1] are in fact sufficient to establish more than the theorems and lemmas in [1] claimed — they actually demonstrate that the closed-loop system relation $(x_1, x_2) \mapsto (e_1, e_2)$ has finite-gain, which implies that in addition to boundedness, the closed-loop mapping is continuous *at the single input point* $(x_1, x_2) = (0, 0)$. Trivially, it follows that when all subsystem positivity properties hold *incrementally*, then the closed-loop mapping is also continuous at any other *constant* input point, i.e., $(x_1(t), x_2(t)) = (x_1(0), x_2(0))\ \forall t$. Whether the mapping is continuous at *non-constant* input points remains unclear *if* the now-standard Zames-Falb [1] line of arguments is to be pursued; in particular, some of the key incremental positivity conditions, which must hold for such arguments to confirm continuity at all points, do not hold in general as our Theorem 1 and Corollary 1 demonstrate. We have also shown that the situation is likewise for Popov and RL/RC multipliers as well. Proofs of the results in this paper can be used to observe trivially that provably the only stability multipliers which preserve incremental positivity of nonlinearities in $\mathcal{N}$ (or $\mathcal{N}_{odd}$) are *constant valued*, i.e. the ones having *frequency independent* Fourier transform. For systems of the form shown in Figure 1, *only* these multipliers may be relied upon to determine continuity as required for stability in the sense of Zames [4]. Of course, *all* the stability multipliers can be reliably used for determining continuity of systems of the form shown in Figure 1 *if* the nonlinearity is restricted to be *linear*.

Of course by taking an entirely different track, it *might be* possible to show that continuity is determined using the stability multipliers. That remains to be validated. Findings in this paper say that until that is done, caution must be exercised in interpreting stability verdict of these multipliers. To be precise, the situation is not ruled out as yet in which although a stability multiplier ratifies a system of the form given by Figure 1 as *stable*, the system actually is critically sensitive to small changes in inputs x_1 and x_2. Such a system is guaranteed to produce a bounded output $y \doteq (y_1, y_2)$ for every bounded *constant* input $x \doteq (x_1, x_2)$ and yet there exists an input perturbation which results in unbounded output y. A physical example of such a perturbation is the time-varying and unpredictable nature of the reference input signal to an auto pilot missile [14]. In the worst case scenario, it may happen that stability analysis performed using stability multipliers may ratify a control law acceptable whereas on implementing it, the missile will fail to adhere to the desired trajectory. It is instructive to note that Fromion *et al* [14] used constant valued multipliers in determining continuity of their examined system, viz. a model of an auto pilot missile governed by a PI controller.

Historically, Lyapunov stability theory and input-output stability theory have developed separately. However, perhaps it would not be out of place to mention that there are systems that are stable in the sense of Lyapunov and yet exhibit a discontinuous behavior. For example, consider a damped pendulum having unit mass and unit length. Let x denote its angular displacement from the hanging down position and let torque u be applied to it. Then equation of motion of the pendulum is $\ddot{x}+g\sin(x)=u$. The system has a stable equilibrium, viz. $x=0$, and an unstable equilibrium, viz. $x=\pi$. Suppose the initial state is $x=0$ and an input u_1 is applied such that the final state is $x=\pi$. Since $x=\pi$ is an unstable equilibrium, applying an input u_2 that differs from u_1 by a vanishingly small amount leads to the final state $x=0$. Thus the system exhibits two vastly different state trajectories whereas difference in the respective inputs is vanishingly small. This demonstrates the existence of a system that is stable in the sense of Lyapunov but not continuous.

IV. Conclusion

We have demonstrated that Popov, Zames-Falb and related dynamical stability multipliers fail to preserve the incremental positivity of monotone nonlinearities. More precisely, we have established that for every monotone nonlinearity N that is not linear, each of these multiplier classes contains a multiplier M such that MN is not incrementally positive. This implies that multiplier stability results should be interpreted with caution since, without incremental positivity, existing stability results cannot establish the continuity property required by Zames' [4] definition of input-output stability.

V. Appendix: Formal Proofs and Key Background Results

First we present two key preliminary results.

Lemma 1: Suppose $N\in\mathcal{N}, x,\bar{x}\in\mathcal{L}_2,\tau\in\mathbb{R},\tau\neq 0$. Let $\widetilde{x}\doteq x-\overline{x}$ and $\widetilde{y}\doteq N(x)-N(\overline{x})$. Then,

$$\text{trace}[r_{\widetilde{y}\widetilde{x}}(0)]\geq 0 \quad \forall x,\bar{x}\in\mathcal{L}_2 \ni \dot{x},\dot{\bar{x}}\in\mathcal{L}_2 \tag{6}$$

Furthermore,

$$\text{trace}[r_{\widetilde{y}\widetilde{x}}(0)-r_{\widetilde{y}\widetilde{x}}(\tau)]\geq 0 \quad \forall x,\bar{x}\in\mathcal{L}_2 \ni \dot{x},\dot{\bar{x}}\in\mathcal{L}_2 \ \forall\tau\in\mathbb{R} \tag{7}$$

holds if and only if N is linear. □

Proof: The inequality (6) follows from (3) and the fact that an $N\in\mathcal{N}$ has a finite incremental gain. It remains to prove that (7) holds if and only if N is linear.

(**if**) Suppose that N is linear. Then for all $x,\overline{x}\in\mathcal{L}_2$,

$$\begin{aligned}\text{trace}[r_{\widetilde{y}\widetilde{x}}(0)-r_{\widetilde{y}\widetilde{x}}(\tau)] &= \langle N(x)-N(\overline{x}),x-\overline{x}\rangle\\ &= \langle N(\widetilde{x}),\widetilde{x}\rangle\\ &\geq 0\end{aligned}$$

where the last inequality follows since N is positive.
(**only if**) Suppose $N\in\mathcal{N}$, N not linear. It needs to be shown that for every such N, there exist $x\in\mathcal{L}_2$ and $\overline{x}\in\mathcal{L}_2$ for which (7) does not hold. Let us first consider the case in which N is SISO.

Assume without loss of generality that N is differentiable. If $N\in\mathcal{N}$ is not linear, then there exist two points $\zeta_1,\zeta_2\in\mathbb{R},\ \zeta_1\neq\zeta_2$ such that

$$\beta\doteq N'(x)\Big|_{x\,=\,\zeta_1} > \alpha\doteq N'(x)\Big|_{x\,=\,\zeta_2} \tag{8}$$

where $\alpha \geq 0$ since $N \in \mathcal{N}$. Existence of signals $x, \overline{x}$ for which (7) fails to hold needs to be demonstrated. For that purpose, choose the test signals $x, \overline{x}$ to be square-wave signal defined as follows.

$$\overline{x}(t) = \begin{cases} \zeta_1 & \text{if } t \in \mathcal{T}_1 \\ \zeta_2 & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} \; ; \; x(t) = \begin{cases} \overline{x}(t) + \epsilon\mu & \text{if } t \in \mathcal{T}_1 \\ \overline{x}(t) + \epsilon & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} \tag{9}$$

where, with $k \in \mathbb{Z}$, $\mathcal{T}_1 \doteq \bigcup_{k=0}^{\ell-1} [2k\delta, (2k+1)\delta)$, and $\mathcal{T}_2 \doteq \bigcup_{k=0}^{\ell-1} [(2k+1)\delta, (2k+2)\delta)$, $\mu = \frac{\alpha+\beta}{2\beta}$; the choices of ϵ and $\ell > 0$ will be explained shortly, $\nu \doteq \delta/\tau > 1, \nu \in \mathcal{Z}$.

For $x, \overline{x}$ as chosen above, $\widetilde{x}$ and $\widetilde{y}$ are given by

$$\widetilde{x}(t) = \begin{cases} \epsilon\mu & \text{if } t \in \mathcal{T}_1 \\ \epsilon & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases} ; \; \widetilde{y}(t) = \begin{cases} \beta\epsilon\mu + O(\epsilon^2) & \text{if } t \in \mathcal{T}_1 \\ \alpha\epsilon + O(\epsilon^2) & \text{if } t \in \mathcal{T}_2 \\ 0 & \text{else.} \end{cases}.$$

Let $a, b, v, u \in \mathbb{R}^\nu$ be defined as $b \doteq [\beta \; \beta \; \ldots \; \beta]^T$, $a \doteq [\alpha \; \alpha \; \ldots \; \alpha]^T$, $v \doteq [\mu \; \mu \; \ldots \; \mu]^T$, $u \doteq [1 \; 1 \; \ldots \; 1]^T$. Let $Q \in \mathbb{R}^{2\ell\nu \times 2\ell\nu}$ have all its entries zero except for the diagonal which is given by $[b^T \; a^T \; \ldots b^T \; a^T]^T$ and the first sub-diagonal which is given by $[-b^T \; -a^T \; \ldots \; -b^T \; -a^T \; \underbrace{-\beta \; -\beta \; \ldots -\beta}_{\nu-1 \text{ terms}}]^T$. Let $w \doteq [v \; u \; \ldots \; v \; u]^T$ $w \in \mathbb{R}^{2\ell\nu}$. It may then be verified that

$$\begin{aligned} &\text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)] = \langle \widetilde{y}, (I - D_\tau)\widetilde{x} \rangle \\ &= \; \epsilon^2 \tau \; w^T Q w \; + \; O(\epsilon^3) \\ &= \; -(\ell - 1)\epsilon^2 \tau [O(1/\ell) + \frac{(\beta - \alpha)^2}{4\beta} + O(\epsilon)]. \end{aligned} \tag{10}$$

Choosing ℓ sufficiently large and ϵ sufficiently small, (10) implies $\text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)] < 0$. This proves the 'only if' part of (7) for the SISO case.

For a MIMO N, the 'only if' part of (7) can be seen to hold as follows. Again, assume without loss of generality that N is differentiable; i.e., the Jacobian matrix $N'(x) = \frac{\partial}{\partial x} N(x)$ exists. Since N is not linear, there exist vectors $\zeta_1, \zeta_2 \in \mathbb{R}^n$ such that the Jacobian matrix of N evaluated at ζ_1 differs from the Jacobian matrix of N evaluated at ζ_2 in at least one entry, say in the (i,j)-th entry. Then, the SISO case arguments, modified by applying the perturbations $\epsilon\mu$ and μ to only the j-th component of $\overline{x}$, may be seen to lead to the desired result. QED.

From Lemma 1, the following result is immediate.

Lemma 2: Suppose $N \in \mathcal{N}$ and $x, \bar{x} \in \mathcal{L}_2$. Let $\widetilde{x} \doteq x - \overline{x}$, $\widetilde{y} \doteq N(x) - N(\overline{x})$. Let $\widetilde{v}(t) \doteq \frac{d}{dt}\widetilde{y}(t) \; \forall t$. Then,

$$\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] \geq 0 \quad \forall x, \overline{x} \in \mathcal{L}_2 \ni \dot{x}, \dot{\overline{x}} \in \mathcal{L}_2 \tag{11}$$

holds if and only if N is linear. □

Proof: **(if)** Suppose that N is linear. First note that

$$\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] = \lim_{\tau \to 0} \frac{\text{trace}[r_{\widetilde{y}\widetilde{x}}(0) - r_{\widetilde{y}\widetilde{x}}(\tau)]}{\tau} \quad \forall x, \overline{x} \in \mathcal{L}_2 \; \forall \tau \in \mathbb{R}. \tag{12}$$

The result then follows using (7).

(only if) Suppose that N is not linear, N SISO. Choose $x, \overline{x} \in \mathcal{L}_2$ as given by (9). Then (10) and (12) imply that

$$\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] = -(\ell - 1)\epsilon^2[O(1/\ell) + \frac{(\beta - \alpha)^2}{4\beta} + O(\epsilon)]. \tag{13}$$

Taking ϵ small enough and ℓ large enough, the result follows. The 'only if' part of (11) for a MIMO N follows as outlined on the lines of the 'only if' part of (7) for a MIMO N. QED.

Proof of Theorem 1: It needs to be proved that if $M \in \mathcal{M}_P$ then MN is incrementally positive for every $N \in \mathcal{N}$ if and only if N is linear. Let $x, \overline{x} \in \mathcal{L}_2$ and write $y \doteq N(x), \overline{y} \doteq N(\overline{x}), \tilde{x} \doteq x - \overline{x}$ with the corresponding $\tilde{y} \doteq y - \overline{y}$.

(if) Suppose that N is linear. In this case, $N(\tilde{x}) = N(x) - N(\overline{x})$. By (4)-(5), we have

$$\langle M\,(N(x) - N(\overline{x}))\,, x - \overline{x}\rangle = \langle MN(\widetilde{x}), \widetilde{x}\rangle \geq 0 \;\forall x, \overline{x} \in \mathcal{L}_2; \tag{14}$$

that is, $M \in \mathcal{M}$ ($M \in \mathcal{M}_{odd}$) is incrementally positivity preserving for all *linear* $N \in \mathcal{N}$ ($N \in \mathcal{N}_{odd}$).

(only if) Now suppose that N is not linear. Consider an $M_* \in \mathcal{M}_P$, its parameter q to be described shortly. Note that

$$\begin{aligned} \langle M^*\,(N(x) - N(\overline{x}))\,, x - \overline{x}\rangle &= \langle \widetilde{y}, \widetilde{x}\rangle + q\,\langle \widetilde{v}, \widetilde{x}\rangle \\ &= \text{trace}[r_{\widetilde{y}\widetilde{x}}(0)] + q\,\text{trace}[r_{\widetilde{v}\widetilde{x}}(0)] \end{aligned} \tag{15}$$

By Lemma 2, there exists $\{x_* \in \mathcal{L}_2, \overline{x}_* \in \mathcal{L}_2, t_* \in \mathbb{R}\}$ such that

$$\gamma_1 \doteq \text{trace}[r_{\widetilde{y}_*\widetilde{x}_*}(0)] \geq 0, \tag{16}$$

$$\gamma_2 \doteq \text{trace}[r_{\widetilde{v}_*\widetilde{x}_*}(0)] < 0 \tag{17}$$

where $\widetilde{x}_* \doteq x_* - \overline{x}_*$ with the corresponding $\widetilde{y}_* \doteq N(x_*) - N(\overline{x}_*)$ and $\widetilde{v}_* \doteq v_* - \overline{v}_*$. Choose the parameter q of the candidate multiplier M_* as

$$q > -\frac{\gamma_1}{\gamma_2}. \tag{18}$$

Note that $q > 0$ so that $M_* \in \mathcal{M}_P$. Also, $\gamma_1 \doteq \text{trace}[r_{\widetilde{y}_*\widetilde{x}_*}(0)]$ is a bounded nonnegative quantity (see (6)). Whence, it follows using (15)–(18) that

$$\begin{aligned} \langle M_*\,(N_*(x_*) - N_*(\overline{x}_*))\,, x_* - \overline{x}_*\rangle &= \gamma_1 + q\gamma_2 \\ &< 0. \end{aligned}$$

QED.

Proof of Corollary 1: Technically, it needs to be proved that given an $M \in \mathcal{M}$ ($M \in \mathcal{M}_{odd}$), MN is incrementally positive for every $N \in \mathcal{N}$ (or, respectively, $M \in \mathcal{N}_{odd}$) if and only if N is linear. It also needs to be proved that given an $M \in \mathcal{M}_{RL}$ (or, alternatively, $M \in \mathcal{M}_{odd}$), MN is incrementally positive for every $N \in \mathcal{N}$ if and only if N is linear. We first prove the result for $\mathcal{M}$.

Let $x, \overline{x} \in \mathcal{L}_2$ and write $y \doteq N(x), \overline{y} \doteq N(\overline{x}), \tilde{x} \doteq x - \overline{x}$ with the corresponding $\tilde{y} \doteq y - \overline{y}$.

(if) Follows on the lines of the proof of the (if) part of Theorem 1.

(only if) Suppose that N is not linear. Consider the multiplier $M : x \mapsto m * x$ with

$$\widehat{m}(j\omega) = \frac{1 + qj\omega}{\epsilon j\omega + 1} \quad \forall\omega \in \mathbb{R}; \quad q, \epsilon > 0. \tag{19}$$

By continuity of inner product (see [15, Chapter 3, pp. 138]),

$$\begin{aligned} \lim_{\epsilon \to 0} \langle M^* (N(x) - N(\overline{x})), x - \overline{x} \rangle &= \langle \widetilde{y}, \widetilde{x} \rangle + q \langle \widetilde{v}, \widetilde{x} \rangle \\ &= \operatorname{trace}[r_{\widetilde{y}\widetilde{x}}(0)] + q \operatorname{trace}[r_{\widetilde{v}\widetilde{x}}(0)]. \end{aligned} \tag{20}$$

Choosing the signals and the parameter q as described in (16)-(18), it follows that for vanishingly small ϵ, the RL multiplier M specified by (19) and (18) fails to preserve incremental positivity of the nonlinearity N. This proves the result for RL multipliers. An RC multiplier is the inverse of an RL multiplier; the result may be established to hold for RC multipliers as well. Since $\mathcal{M}_{RL} \subset \mathcal{M} \subset \mathcal{M}_{odd}$ [7], the result also holds for Zames-Falb multipliers $\mathcal{M}$ and $\mathcal{M}_{odd}$. QED.

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