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Rational Multiplier IQC's for Uncertain Time-delays and LMI Stability Conditions[1]

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Abstract

This paper describes a set of rational delay-dependent IQC's for time-delay uncertainty. The set is linearly parameterized in terms of the frequency-response of a complex-valued multiplier. Furthermore, this paper provides a set of non-frequency dependent LMI conditions for stability test by applying finite frequency positive real condition by [5].

1 Introduction

Stability analysis of time-delayed systems has been studied with great interest during last decades [2], [8]. Stability criteria for time-delay systems tend to fall into one of two categories according to their dependence upon delay size: *delay-dependent* or *delay-independent*. Delay-independent criteria provide conditions for stability without regard for the size of time delays. Delay-dependent stability criteria are concerned with the size of delays and usually provide upper bounds of time delays for which one can guarantee the stability of the system.

The robust stability methodology is useful in dealing with structured uncertainties (see [1]). Time-delays can be considered as structured uncertainties and can be analyzed using robust control theories[12]. Many of methods that have been developed within the area of robust control have been reformulated within the framework of integral quadratic constraints (IQC's)[9]. Fu *et al.*[3] and Jun *et al.*[6] provided delay-dependent results for robust stability using IQC approach and linear matrix inequalities (LMI's), which can provide an estimate of delay margin. There are methods which can calculate the exact delay bound for a given system. However, those algorithms cannot be extended readily if other uncertainties in the system is considered while ones by using IQC approach can be.

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A key to successful application of IQC stability methods is discovery of suitable class of IQC's for various classes of uncertainties. Jun *et al.*[7] and Scorletti[12] presented a broad class of IQC's for time delays which are linearly parameterized in terms of a complex multiplier $M(j\omega)$ and two real multipliers, respectively. They have shown that the existing IQC's for time-delay such as [9] corresponds to such particular cases with particular choices of the multiplier, which means that their multiplier-parameterized IQC's will generally produce better results.

The difficulty in checking the stability conditions in [7] and [12] for a given time-delay system is that the condition should be satisfied at each frequency $\omega \in \mathbb{R}$, which makes the problem infinite dimensional LMI problem. One possible solution to this problem is to compute it at each frequency of a preselected frequency grid since it is convex optimization problem involving complex-valued LMI problem for a given time-delay $\bar{\tau}$ at each frequency ω. However, there is crucial drawback in this approach: there exists a possibility that the critical frequency where the condition does not hold is missed in the frequency grid. If the frequency grid is not tight enough, this approach is always open to this problem. The computation, however, gets too expensive if the frequency is tight enough, which renders this method impractical. A popular method used in solving a frequency dependent LMI problem is to convert it to a frequency independent LMI problem by applying the Kalman-Yakubovich-Popov (KYP) Lemma. However, the KYP Lemma is not applicable directly in solving the stability condition in [7] and [12] since it consists of two separate IQC's on two distinct frequency intervals.

Another practical difficulty in application of the results in [7] and [12] lies in the fact that both their IQC's and multiplier $M(j\omega)$ are non-rational matrix functions of ω. The trouble caused by the non-rationality of the multiplier $M(j\omega)$ can be cured easily by restricting the class of complex-valued matrix multipliers $M(j\omega)$ to the set of fixed-order real rational matrix functions with $\mathrm{herm}(M(j\omega)) > 0$. And the non-rationality in the IQC's can be resolved by finding a rational approximation of them.

Table 1: Notation

Symbol	Meaning
$\mathbb{R}$, $(\mathbb{R}_+)$	Set of all (positive) real numbers
$\mathbb{C}$	Set of all complex numbers
$A(s)^*$	$A(-s)^T$, conjugate transpose
A^{-*}	$= (A^*)^{-1}$
I_q	$q \times q$ identity matrix
$\circ$	Convolution operator
$\text{herm}(m)$	$= \frac{1}{2}(m + m^*)$
$\text{skew}(m)$	$= \frac{1}{2}(m - m^*)$
$\mathfrak{Re}(\cdot)$	Real part of $(\cdot)$
$\mathfrak{Im}(\cdot)$	Imaginary part of $(\cdot)$
$\hat{x}(j\omega)$	Fourier transform of the signal $x(t)$
$< x, y >$	$= \int_{-\infty}^{\infty} y(t)^T x(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(j\omega)^* \hat{x}(j\omega)d\omega$
$\|\|x\|\|$	$= \sqrt{< x, x >}$

In this paper, we will first provide two distinct rational multiplier IQC's for time-delay uncertainty on two distinct frequency range and then present a non-frequency dependent LMI stability condition by using the recent *finite frequency strictly positive real condition* by T. Iwasaki *et al.* [5], which extends the KYP Lemma to test strict positive realness on a finite frequency range.

This paper is organized as follows: preliminary background is covered in Section 2 and problem is formulated in Section 3. Our main results is given in Section 4. Finally conclusions are in Section 5.

2 Preliminaries

This section briefly covers preliminary results such as IQC Theorem, multiplier IQC's for time-delay by M. Jun *et al.* [7] and finite frequency positive real condition by T. Iwasaki *et al.* [5]. Consider the feedback system in Figure 1 where G and Δ are bounded causal operators on $\mathcal{L}_{2e}^m[0,\infty)$ and $\mathcal{L}_{2e}^l[0,\infty)$, respectively.

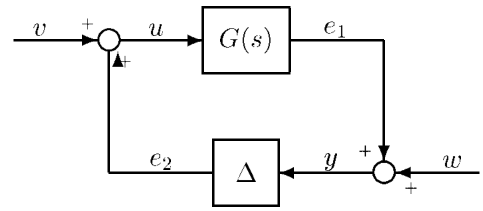

Figure 1: Basic feedback configuration

Definition 1 *(cf. [9]) The interconnection G and Δ is said to be* well-posed *if the map $(y,u) \mapsto (v,w)$ has a causal inverse on $\mathcal{L}_{2e}^{l+m}[0,\infty)$. The feedback system is said to be* stable *if it is well-posed and inputs $v \in \mathcal{L}_2^m[0,\infty), w \in \mathcal{L}_2^l[0,\infty)$ lead to outputs $e_1, y \in \mathcal{L}_2^l[0,\infty)$ and $e_2, u \in \mathcal{L}_2^m[0,\infty)$. If, in addition, there exists a constant $C > 0$ such that*

$$\int_0^T (|y|^2 + |u|^2)dt \le C \int_0^T (|v|^2 + |w|^2)dt, \quad \forall\, T \ge 0,$$

then, the system is said to be stable with finite gain.

Definition 2 *[9, pp.820] Let $\Pi : j\mathbb{R} \mapsto \mathbb{C}^{(l+m)\times(l+m)}$ be a measurable Hermitian valued function. A bounded operator $\Delta : \mathcal{L}_{2e}^l[0,\infty) \mapsto \mathcal{L}_{2e}^m[0,\infty)$ is said to* satisfy the IQC defined by Π, *if*

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\Delta}(y)(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\Delta}(y)(j\omega) \end{bmatrix} d\omega \ge 0 \qquad (1)$$

for all $y \in \mathcal{L}_2^l[0,\infty)$.

IQC Theorem *(cf. [9, Theorem 1]) Assume that:*

i) for every $\alpha \in [0,1]$, the interconnection of G and Δ_α is well-posed where Δ_α is a parameterization of Δ which satisfies

a) $\Delta = \Delta_\alpha\big|_{\alpha=1}$,

b) Δ_α is bounded and causal for $\alpha \in [0,1]$,

c) there exists $\gamma > 0$ such that

$$||\Delta_{\alpha_1}(y) - \Delta_{\alpha_2}(y)|| \le \gamma|\alpha_1 - \alpha_2| \cdot ||y||$$

for all $\alpha_1, \alpha_2 \in [0,1]$,

ii) the interconnection of G and $\Delta_\alpha\big|_{\alpha=0}$ is stable with finite gain,

iii) for every $\alpha \in [0,1]$, the IQC defined by Π is satisfied by Δ_α, that is,

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\Delta}(y)(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \widehat{\Delta}(y)(j\omega) \end{bmatrix} d\omega \ge 0 \qquad (2)$$

iv) there exists $\epsilon > 0$ such that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \le -\epsilon I, \quad \forall\, \omega \in \mathbb{R}. \qquad (3)$$

Then, the feedback interconnection of G and Δ is stable with finite gain for all $\alpha \in [0,1]$.

The multiplier IQC's for time-delay uncertainty by M. Jun *et al.* [7] is stated as follows:

Theorem 1 *[7] Let $\Delta(j\omega) \triangleq \delta(j\omega)I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \le \tau \le \bar{\tau}$. Then, $\Delta(j\omega)$ satisfies the IQC (1) defined by*

$$\Pi_M(j\omega) \triangleq \begin{cases} herm\left(\begin{bmatrix} I_q \\ -I_q \end{bmatrix} M(j\omega)\begin{bmatrix} -e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q \end{bmatrix}\right), & |\omega| < \frac{2\pi}{\bar{\tau}} \\ herm\left(\begin{bmatrix} M(j\omega) & 0 \\ 0 & -M(j\omega) \end{bmatrix}\right), & |\omega| \ge \frac{2\pi}{\bar{\tau}} \end{cases} \qquad (4)$$

where $M(j\omega) \in \mathbb{C}^{q\times q}$ and $herm(M(j\omega)) \ge 0$ for all $|\omega| < \frac{2\pi}{\bar{\tau}}$.

The definition for finite frequency strictly positive real transfer function can be stated as follows:

Definition 3 *[5] A stable square transfer function $G(s)$ is said to be* finite frequency strictly positive real (FFSPR) *with bandwidth $\bar{\omega}$ if*

$$G(j\omega) + G(j\omega)^* > 0, \quad \forall\, |\omega| \leq \bar{\omega}. \tag{5}$$

The following lemma is a generalization of the KYP Lemma where a frequency domain condition is required to hole only for a given finite frequency range.

Lemma 1 *[5] Let a real scalar $\omega_0 > 0$, matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$ and a Hermitian matrix $\Pi \in \mathbb{C}^{(n+m)\times(n+m)}$ be given. Suppose A has no eigenvalues on the imaginary axis. Then the following statements are equivalent:*

i) For all ω such that $|\omega| \leq \omega_0$,

$$\begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix}^* \Pi \begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix} < 0. \tag{6}$$

ii) There exist Hermitian matrices $P \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{n\times n}$ such that $Q > 0$ and

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}-Q & P \\ P & \omega_0^2 Q\end{bmatrix} \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Pi < 0. \tag{7}$$

3 Problem Formulation

First, we define a class of multipliers, $\mathcal{M}_r^{(n)}$ to be set of fixed order LTI transfer matrices which satisfy the followings:

Definition 4 *Given q and ω_0, an LTI transfer matrix $M(s)$ can be said to belong to the class $\mathcal{M}_r^{(n)}(q, \omega_0)$ if $M(s)$ satisfies* [1]

$$M(s) = \sum_{j=0}^{n} A_j s^j, \quad A_j \in \mathbb{R}^{q\times q}, \ j = 1, \cdots n, \tag{8}$$

$$M(j\omega) + M(j\omega)^* > 0, \quad \forall\, |\omega| < \omega_0. \tag{9}$$

It can be noticed that each multiplier $M(s) \in \mathcal{M}_r^{(n)}$ can be decomposed by frequency dependent transfer matrix part and frequency independent parameter part, viz., $M(s) = \Psi(s)^* H \Psi(s)$. For example, if $M(s) = A_1 s + A_0$ where $A_1, A_0 \in \mathbb{R}^{q\times q}$, then it can be expressed by

$$M(s) = \begin{bmatrix}-sI_q & I_q\end{bmatrix} \begin{bmatrix}0 & 0 \\ A_1 & A_0\end{bmatrix} \begin{bmatrix}sI_q \\ I_q\end{bmatrix} = \Psi(s)^* H \Psi(s).$$

By restricting the class of multiplier $M(j\omega)$ in Theorem 1 to $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$, we get a rational approximation of $\Pi_M(j\omega)$ for $|\omega| \geq 2\pi/\bar{\tau}$. Then, our problem to be solved is to find a rational approximation of $\Pi_M(j\omega)$ for $|\omega| < 2\pi/\bar{\tau}$ and find finite dimensional LMI stability conditions. They can be stated as follows:

Problem 1 *Find a rational approximation of the following:*

$$herm\left(\begin{bmatrix}I_q \\ -I_q\end{bmatrix} M_r(j\omega) \begin{bmatrix}-e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q\end{bmatrix}\right) \tag{10}$$

which defines IQC (1) satisfied by $\Delta(j\omega) = e^{-j\omega\tau} I_q$ for $|\omega| < 2\pi/\bar{\tau}$ and $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$.

Problem 2 *With rational versions of multiplier IQC in Theorem 1 on two separate frequency range, find finite dimensional LMI condition for stability of a given time-delay system.*

4 Main Result

4.1 Rational Multiplier IQC

Let $\begin{bmatrix}\tilde{y}(j\omega) \\ \tilde{u}(j\omega)\end{bmatrix} \triangleq S_r(j\omega) \begin{bmatrix}y(j\omega) \\ u(j\omega)\end{bmatrix}$ where $S_r(j\omega)$ is invertible, let $u(j\omega) = \Delta(j\omega) y(j\omega)$, and let $\tilde{u}(j\omega) = \widetilde{\Delta}(j\omega)\tilde{y}(j\omega)$. $S_r(j\omega)$ is called *sector transformation matrix* (See [4]). The non-rationality of Eq. (10) is due to the exponential function that was originated from non-rational sector transformation matrix. So, our first step is to find a rational sector transformation matrix $S_r(j\omega)$ that can lead to the solution of Problem 1. The following lemma proposes a generalization of the sector transformation matrix in [7] and opens the way to find a rational version of $\Pi_M(j\omega)$.

Lemma 2 *Let $p(j\omega) = p^*(-j\omega)$ $\forall\, \omega$ and let the transformation matrix $S_r(j\omega) \in \mathbb{C}^{2\times 2}$ be*

$$S_r(j\omega) = \begin{bmatrix}S_{11}(j\omega) & S_{12}(j\omega) \\ S_{21}(j\omega) & S_{22}(j\omega)\end{bmatrix} = \begin{bmatrix}p(j\omega) & -p^*(j\omega) \\ -1 & 1\end{bmatrix} \tag{11}$$

where $p(j\omega)$ is a scalar complex-valued function such that $-\pi < \measuredangle p(j\omega) \leq -\omega$ for $0 < \omega < \pi$. If $\Delta(j\omega) \in \{\, \Delta \mid \Delta = e^{-j\varphi} \,\}$ where $\varphi \in (0, 2\omega)$ when $0 < \omega < \pi$, then $\widetilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$.

[1] We may w.l.o.g. confine our attention to real *polynomial* multipliers [11]. When it is needed to work with proper or strictly proper transfer function $M(s)$, one may always substitute for $M(s)$ the equivalent proper or strictly proper multiplier

$$\tilde{M}(s) = \frac{1}{d(s)d(s)^*} M(s)$$

where $d(s)$ is *any* scalar polynomial of degree at least half that of $M(s)$.

Proof:

$$\widetilde{\Delta}(j\omega) = \frac{\tilde{u}(j\omega)}{\tilde{y}(j\omega)} = \frac{S_{21}(j\omega)y(j\omega) + S_{22}(j\omega)u(j\omega)}{S_{11}(j\omega)y(j\omega) + S_{12}(j\omega)u(j\omega)}$$
$$= \frac{-(1-\Delta(j\omega))}{\Re\mathfrak{e}(p(j\omega))(1-\Delta(j\omega)) + \Im\mathfrak{m}(p(j\omega))(j + j\Delta(j\omega))}$$
$$= -\frac{1}{\Im\mathfrak{m}(p(j\omega))} \cdot \frac{1}{\frac{\Re\mathfrak{e}(p(j\omega))}{\Im\mathfrak{m}(p(j\omega))} + \frac{j + j\Delta(j\omega)}{1-\Delta(j\omega)}}$$

since $\Delta(j\omega) \neq 1$ and $\Im\mathfrak{m}(p(j\omega)) \neq 0$. The fact $\Delta(j\omega) = e^{-j\varphi} = \cos\varphi - j\sin\varphi$ leads to

$$\widetilde{\Delta}(j\omega) = -\frac{1}{\Im\mathfrak{m}(p(j\omega))} \cdot \frac{1}{\cot(\measuredangle p(j\omega_*)) + \cot(\varphi/2)} \geq 0$$

since $\cot(\measuredangle p(j\omega_*)) + \cot(\varphi/2) \geq 0$ and $\Im\mathfrak{m}(p(j\omega)) < 0$. This comes from the fact that $\cot(\cdot)$ is strictly decreasing and $-\pi < \measuredangle p(j\omega) \leq -\omega < -\varphi/2 < 0$ when $0 < \omega < \pi$. ∎

Comparing with the $S(j\omega)$ in [7], it can be noticed that it is the case when $p(j\bar{\tau}\omega/2) = e^{-j\bar{\tau}\omega/2}$ and $p(j\bar{\tau}\omega/2) = e^{-j\bar{\tau}\omega/2}$ satisfies all conditions in Lemma 2. By Lemma 2, we can find a rational sector transformation $S_r(j\omega)$ and a rational version $\Pi_{M_r}(j\omega)$ for time-delay uncertainty with the rational transformation matrix $S_r(j\omega)$ if we restrict the scope of the function $p(s)$ to be a real rational function.

Theorem 2 *Let $\Delta(j\omega) \triangleq \delta(j\omega)I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \leq \tau \leq \bar{\tau}$. Suppose that $p(j\omega)$ is a scalar real rational function such that*

i) $\measuredangle p(j0) = 0$ and $\measuredangle p(j\pi) = -\pi$,

ii) $-\pi < \measuredangle p(j\omega) \leq -\omega,\ \ 0 < \omega < \pi$.

Then, $\Delta(j\omega)$ satisfies the IQC defined by

$$\Pi_{M_r}(j\omega) \triangleq \begin{cases} herm\left(\begin{bmatrix} -I_q \\ I_q \end{bmatrix} M_r(j\omega)\begin{bmatrix} p(j\frac{\bar{\tau}\omega}{2})I_q & -p(j\frac{\bar{\tau}\omega}{2})^* I_q \end{bmatrix}\right), & |\omega| < \frac{2\pi}{\bar{\tau}} \\ herm\left(\begin{bmatrix} M_r(j\omega) & 0 \\ 0 & -M_r(j\omega) \end{bmatrix}\right), & |\omega| \geq \frac{2\pi}{\bar{\tau}} \end{cases} \quad (12)$$

where $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$.

Proof: For simplicity, only the case when $\omega \geq 0$ is considered. It is proved similarly when $\omega < 0$.

It is trivial to check when $\omega = 0$. Next, when $0 < \omega < \frac{2\pi}{\bar{\tau}}$, it can be noticed that

$$\text{herm}\left(\begin{bmatrix} -I_q \\ I_q \end{bmatrix} M_r(j\omega)\left[p(j\tfrac{\bar{\tau}\omega}{2})I_q \quad -p(-j\tfrac{\bar{\tau}\omega}{2})I_q\right]\right)$$
$$= S_r(j\omega)^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} S_r(j\omega)$$

where $S_r(j\omega)$ is defined by

$$S_r(j\omega) \triangleq \begin{bmatrix} p(j\bar{\tau}\omega/2)I_q & -p(-j\bar{\tau}\omega/2)I_q \\ -I_q & I_q \end{bmatrix}.$$

Also it can be said by Lemma 2 that

$$\begin{bmatrix} I_q \\ \widetilde{\Delta}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M_r j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} \begin{bmatrix} I_q \\ \widetilde{\Delta}(j\omega) \end{bmatrix}$$
$$= \frac{1}{\cot(\measuredangle p(j\omega\bar{\tau}/2)) + \cot(\omega\tau/2)} \cdot \text{herm}(M_r(j\omega)) \geq 0$$

since $\text{herm}(M_r(j\omega)) \geq 0$ by assumption. This leads to

$$0 \leq \begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix}$$
$$= \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}^* S_r(j\omega)^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} S_r(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}$$
$$= \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}.$$

Now, let us consider the case when $\omega \geq \frac{2\pi}{\bar{\tau}}$. We have

$$\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}$$
$$= \text{herm}(M_r(j\omega)) - \Delta(j\omega)^* \text{herm}(M_r(j\omega))\Delta(j\omega)$$
$$= \text{herm}(M_r(j\omega))\left(I_q - \Delta(j\omega)^*\Delta(j\omega)\right) = 0.$$

∎

The next problem is how to find a transfer function $p(s)$ which satisfies the phase requirements. There are myriad of rational or polynomial transfer function $p(s)$ such that the conditions i) and ii) in Theorem 2 are satisfied. Here, we propose an easy method to find such a $p(s)$ using a rational approximation of e^{-s}. We will use the following fact:

Proposition 1 *Let $f(s) \triangleq \dfrac{(1-\frac{1}{2n}s)^n}{(1+\frac{1}{2n}s)^n}$, the n^{th} order rational approximation of e^{-s}. Then, $-\omega \leq \measuredangle f(j\omega) \leq 0$ for $0 \leq \omega \leq \pi$ and $\measuredangle f(j0) = 0$. Furthermore, $g(\omega) \triangleq \omega + \measuredangle f(j\omega)$ is monotonically increasing function of ω.*

Proof: Omitted. ∎

The Proposition 1 says that the phase of $f(s)$ always leads the phase of e^{-s}. Therefore, the main idea in getting an appropriate $p(s)$ is to put more delay to $f(s)$, viz., define $p(s)$ to be $p(s) = \dfrac{(1-\frac{1}{2n}\alpha s)^n}{(1+\frac{1}{2n}\alpha s)^n}$. The value of α is determined so that the phase of $p(j\pi)$ is same as that of $e^{-j\pi}$. Then, the next theorem stipulates that such $p(s)$ satisfies the requirements i) and ii) in Theorem 2.

Proposition 2 *Let $p(s) \triangleq \dfrac{(1-\frac{1}{2n}\alpha s)^n}{(1+\frac{1}{2n}\alpha s)^n}$ where α is defined as*

$$\alpha \triangleq \frac{2n}{\pi}\tan\frac{\pi}{2n}. \quad (13)$$

Then, $p(s)$ satisfies the requirements i) and ii) in Theorem 2.

Proof: Omitted. ∎

4.2 Finite Frequency Passivity and Non-Frequency Dependent LMI Formulation

By applying IQC Theorem in conjunction with $\Pi_M(j\omega)$ in Theorem 1, the stability condition of a given time-delay system can be stated as follows [7]:

Stability Criterion for Time-delay Systems *For a given $\bar{\tau} > 0$ suppose that the time-delay uncertainty is expressed as $\Delta(j\omega) \triangleq \delta(j\omega)I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \le \tau \le \bar{\tau}$. Let $\dot{w} \in \mathcal{L}_{2e}^q[0,\infty)$. Assume that:*

i) the feedback system is finite-gain stable when $\tau = 0$,

ii) $G(s)$ is strictly proper,

iii) there exists an $M(j\omega)$ with $herm(M(j\omega)) \ge 0$ for $|\omega| < \frac{2\pi}{\bar{\tau}}$ such that for some $\tilde{\epsilon} > 0$

$$\begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix}^* \Pi_M(j\omega) \begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix} \le -\tilde{\epsilon} I, \quad \forall\, \omega \in \mathbb{R}, \tag{14}$$

Then, the feedback system in Figure 1 is robust against the time-delay $0 \le \tau \le \bar{\tau}$ in the sense that the mapping $(v, w, \dot{w}) \mapsto (u, y)$ is finite-gain stable. □

As pointed out, it is not easy to check the condition (14) without frequency gridding even if we use $\Pi_{M_r}(j\omega)$ in Eq. (12) instead of $\Pi_M(j\omega)$. Therefore it is needed to convert frequency dependent LMI condition (14) to non-frequency dependent LMI condition. In this section, we provide a non-frequency dependent finite dimensional LMI condition for stability of time-delay system by using the results in Section 4.1 and by applying Lemma 1.

First, we define some variables needed for our LMI formulation. As mentioned in Section 3, the multiplier $M_r(j\omega) \in \mathcal{M}_r^{(n)}$ can be decomposed into $M_r(j\omega) = \Psi(j\omega)^* H \Psi(j\omega)$ where herm$(H) > 0$. We define A_1, B_1, C_1 and D_1 to be

$$\begin{bmatrix} \Psi(s) & 0 \\ 0 & \Psi(s) \end{bmatrix} S_r(s) \begin{bmatrix} G(s) \\ I \end{bmatrix} \triangleq C_1(sI - A_1)^{-1}B_1 + D_1.^2 \tag{15}$$

Let $\hat{G}(\hat{s}) \triangleq G(1/s)$ and assume that there exists a state-space realization of $\hat{G}(\hat{s})$. Define A_2, B_2, C_2 and D_2 to be

$$\begin{bmatrix} \hat{G}(\hat{s}) \\ I \end{bmatrix} \triangleq C_2(\hat{s}I - A_2)^{-1}B_2 + D_2. \tag{16}$$

Then the frequency dependent LMI condition for stability (14) can be transformed to non-frequency dependent LMI condition with rational approximation.

[2] If $\left[\begin{smallmatrix} \Psi(s) & 0 \\ 0 & \Psi(s) \end{smallmatrix}\right] S_r(s) \left[\begin{smallmatrix} G(s) \\ I \end{smallmatrix}\right]$ is not proper or strictly proper, multiply the reciprocal of any scalar polynomial $d(s)$ which makes $\frac{1}{d(s)} \left[\begin{smallmatrix} \Psi(s) & 0 \\ 0 & \Psi(s) \end{smallmatrix}\right] S_r(s) \left[\begin{smallmatrix} G(s) \\ I \end{smallmatrix}\right]$ proper or strictly proper and then find a state-space realization of it.

Theorem 3 *For a given $\bar{\tau} > 0$ suppose that the time-delay uncertainty is expressed as $\Delta(j\omega) \triangleq \delta(j\omega)I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \le \tau \le \bar{\tau}$. Let $\dot{w} \in \mathcal{L}_{2e}^q[0,\infty)$. Assume that:*

i) the feedback system is finite-gain stable when $\tau = 0$,

ii) $G(s)$ is strictly proper,

iii) $\hat{G}(\hat{s}) = G(1/s)$ is proper,

iv) there exists an Q_1, Q_2, H and M such that

$$\begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D_1 \end{bmatrix}^* \begin{bmatrix} -Q_1 & P_1 & 0 \\ P_1 & \left(\frac{2\pi}{\bar{\tau}}\right)^2 Q_1 & 0 \\ 0 & 0 & \left[\begin{smallmatrix} 0 & H^* \\ H & 0 \end{smallmatrix}\right] \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D_1 \end{bmatrix} < 0, \tag{17}$$

$$\begin{bmatrix} A_2 & B_2 \\ I & 0 \\ C_2 & D_2 \end{bmatrix}^* \begin{bmatrix} -Q_2 & P_2 & 0 \\ P_2 & \left(\frac{\bar{\tau}}{2\pi}\right)^2 Q_2 & 0 \\ 0 & 0 & \left[\begin{smallmatrix} M & 0 \\ 0 & -M \end{smallmatrix}\right] \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ I & 0 \\ C_2 & D_2 \end{bmatrix} < 0, \tag{18}$$

$$Q_1 > 0, \;\; Q_2 > 0, \;\; H + H^* > 0, \;\; M = M^* > 0. \tag{19}$$

where A_i, B_i, C_i and D_i, $i = 1, 2$ are defined as in Eq. (15) and Eq. (16).

Then, the feedback system in Figure 1 is robust against the time-delay $0 \le \tau \le \bar{\tau}$ in the sense that the mapping $(v, w, \dot{w}) \mapsto (u, y)$ is finite-gain stable.

Proof: By IQC Theorem, we just have to show that the statement

> *There exists an $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$ such that for some $\tilde{\epsilon} > 0$*
>
> $$\begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix} \le -\tilde{\epsilon} I, \quad \forall\, \omega \in \mathbb{R},$$

is equivalent to the condition iii).

First, consider the stability condition when $|\omega| < 2\pi/\bar{\tau}$. By Theorem 2 and Eq. (14), the stability condition on this frequency range can be stated as

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* S_r(j\omega)^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} S_r(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \tag{20}$$

for $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$. Since $M(j\omega)$ can be decomposed into $M(j\omega) = \Psi(j\omega)^* H \Psi(j\omega)$, the Eq. (20) is equivalent to saying that

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* S_r(j\omega)^* \begin{bmatrix} \Psi(j\omega) & 0 \\ 0 & \Psi(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & H^* \\ H & 0 \end{bmatrix} \times \begin{bmatrix} \Psi(j\omega) & 0 \\ 0 & \Psi(j\omega) \end{bmatrix} S_r(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \tag{21}$$

$$H + H^* > 0. \tag{22}$$

Therefore, by Lemma 1, the frequency dependent LMI (21) and (22) are equivalent to the frequency independent LMI conditions Eq. (17), $Q_1 > 0$, and $H + H^* > 0$.

Next, let us consider stability condition for $|\omega| \geq 2\pi/\bar{\tau}$. We can notice that the condition

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad |\omega| \geq \frac{2\pi}{\bar{\tau}}$$

is equivalent to the condition

$$\begin{bmatrix} \hat{G}(j\hat{\omega}) \\ I \end{bmatrix}^* \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} \hat{G}(j\hat{\omega}) \\ I \end{bmatrix} < 0, \quad |\hat{\omega}| \leq \frac{\bar{\tau}}{2\pi}. \quad (23)$$

Thus, by Lemma 1, the condition (23) is equivalent to frequency independent LMI conditions Eq. (18), $Q_2 > 0$, and $M = M^* > 0$. ■

The properness condition of $\hat{G}(\hat{s})$ is not a big problem. In practice, we can add an additional parameter $\epsilon \ll 1$ to make it proper when $\hat{G}(\hat{s})$ is not proper. For example, when $G(s) = 1/s$, then, define $\hat{G}(\hat{s}) \triangleq \hat{s}/(\epsilon\hat{s} + 1)$.

When synthesis problem is considered, a controller $K(s) = C_K(sI - A_K)^{-1}B_K + D_K$ is inserted into the general feedback configuration in Figure 1 and synthesis problem becomes similar to μ synthesis problem. This is BMI problem [10], not LMI problem, since the LMI's in Theorem 3 is affine in M, H, Q_1, Q_2 and controller parameters(A_K, B_K, C_K, D_K) respectively but not jointly. Still, we may use suboptimal *D-K iteration-like* method, viz., first set initial controller parameters A_K, B_K, C_K and D_K, and then solve LMI's in Theorem 3 with variables M, H, Q_1 and Q_2. If LMI's are not feasible, fix the variables M, H, Q_1, Q_2 with the values which yield minimum and then solve the LMI's with variables A_K, B_K, C_K and D_K. Repeat this iteration until the LMI's are feasible with desired time-delay $\bar{\tau}$ is obtained.

EXAMPLE 1
Consider the following autonomous system

$$\dot{x}(t) = A_0 x(t) + A_d x(t - \tau), \qquad 0 \leq \tau \leq \bar{\tau}$$

with

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & 0.25 \end{bmatrix}, \; A_d = \begin{bmatrix} -1 & 0 \\ -0.1 & -0.85 \end{bmatrix} \qquad (24)$$

which is the same example considered in Fu *et al.* [3]. We use the second order transfer function $p(s) = \frac{0.1013s^2 - 0.6366s + 1}{0.1013s^2 + 0.6366s + 1}$ produced by Proposition 2. The estimate of maximum delay margin using Theorem 1 is $\tau_{max} = 1.489$. Simulation was performed with the MATLAB LMI Control Toolbox by the MathWorks.

As comparisons, the estimate of allowable maximum time delay is $\tau_{max} = 0.6417$ when the Theorem 6 in [3] is applied and it is $\tau_{max} = 0.9848$ when the Theorem 7 in [3] is applied. It is $\tau_{max} = 0.9999$ with application of LMI condition by [6] while the optimal value for the system with the given parameters, A_0 and A_d, is $\tau_{opt} = 1.54$ [3]. We can see that our result is less conservative than [3] and [6]. □

5 Conclusion

In an effort to find a rational version of $\Pi_M(j\omega)$ in [7], we provide a generalized sector transformation matrix $S_r(j\omega)$ which transforms an arc to positive real ray or line segment in complex plane. The $S_r(j\omega)$ is parameterized in terms of scalar complex-valued function $p(j\omega)$ which satisfies a certain phase conditions. We also provide an easy algorithm to find one of such $p(j\omega)$'s using a rational approximation of e^{-s} and a class of rational $\Pi(j\omega)$'s which define the IQC satisfied by time-delay uncertainty which are linearly parameterized in terms of a real rational multiplier $M_r(j\omega)$. Finally, we present a set of non-frequency dependent LMI's to check the robust stability of time-delay systems with application of finite frequency positive real condition by [5], which opens the way to delay-dependent robust controller synthesis via *D-K* iteration-like technique.

References

[1] J. C. Doyle. Analysis of feedback systems with structured uncertainties. *IEE Proc.*, 129-D(6):242–250, November 1982.

[2] L. Dugard and E. I. Verriest. *Stability and Control of Time-delay Systems*, volume 228 of *Lecture Notes in Control and Information Science*. Springer, New York, NY, 1998.

[3] M. Fu, H. Li, and S. I. Niculescu. *Robust stability and stabilization of time-delay systems via integral quadratic constraint approach*, volume 228 of *Lecture Notes in Control and Information Science*, chapter 4. Springer, 1998.

[4] K. C. Goh and M. G. Safonov. Robust analysis, sectors, and quadratic functionals. In *Proc. of IEEE Conf. on Decision and Control*, pages 1988–1993, New Orleans, LA, 1995.

[5] T. Iwasaki, S. Hara, and H. Yamauchi. Structure/control design integration with finite frequency positive real property. In *Proc. of the American Control Conference*, pages 549–553, Chicago, IL, 2000.

[6] M. Jun and M. G. Safonov. Stability analysis of a system with time-delayed states. In *Proc. of the American Control Conference*, pages 949–952, Chicago, IL, June 2000.

[7] M. Jun and M. G. Safonov. Multiplier IQCs for uncertain time-delays. *J. of the Franklin Institute*, 338:335–351, March 2001.

[8] M. Malek-Zavarei and M. Jamshidi. *Time-delay Systems Analysis, Optimization and Applications*. North Holland, Amsterdam, 1987.

[9] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. on Auto. Control*, AC-42(6):819–830, June 1997.

[10] M. G. Safonov, K. C. Goh, and J. H. Ly. Control system synthesis problem via bilinear matrix inequalities. In *Proc. of the American Control Conference*, pages 45–49, Baltimore, MD, 1994.

[11] M. G. Safonov and R. Y.Chiang. *Real/complex k_m-synthesis without curve fitting*, volume 56 of *Control and Dynamic Systems*, pages 303–324. Academic Press, New York, NY, 1993.

[12] G. Scorletti. Robustness analysis with time-delay. In *Proc. of IEEE Conf. on Decision and Control*, pages 3824–3829, San Diego, CA, December 1997.