Rational Multiplier IQC's for Uncertain Time-delays and LMI Stability Conditions

Myungsoo Jun and Michael G. Safonov

Abstract— **This paper describes a set of delay-dependent IQC stability conditions for time-delay uncertainty. The IQC's are linearly parameterized in terms of a pair of rational stability multipliers, each active over one of a pair of complementary frequency intervals. Using the finite-frequency positive real lemma, each of these finite-frequency IQC conditions are shown to be equivalent to a frequency-independent LMI condition, thereby dispensing with the need for frequency-sweeping.**

Keywords— **Time-delay system, stability criteria, multiplier, robust control, IQC.**

I. Introduction

The robust stability methodology is useful in dealing with structured uncertainties [1], [2]. In recent years, robust control theory has been reformulated within the framework of integral quadratic constraints (IQC's) [3], which in turn are linked via the Kalman-Yakubovich-Popov (KYP) lemma to Linear Matrix Inequalities (LMI's). A salient feature of the IQC stability results is that they apply directly to complex interconnected systems consisting of any number of different types of IQC bounded uncertainties. Key to minimizing the conservativeness of robustness results based on IQC/LMI stability theory is the discovery of linear parameterizations of the broadest possible classes of IQC's for each type of uncertainty (cf. [4]).

Time-delays have been considered as a type of structured uncertainty for analysis using robust control techniques [5], [6]. Megretski *et al.* [3], Fu *et al.* [7] and Jun *et al.* [6] provided *delay-dependent* results based on IQC's and LMI's. Scorletti [5] and Jun *et al.* [8] expanded these results, determining the broadest available class of IQC's for time delays, linearly parameterized in terms of a positive-real frequency-dependent multiplier matrix. The results of [5] and [8] involve 'switching multipliers'. That is, a frequency-dependent multiplier matrix makes a typically nonsmooth change from one complex frequency-dependent multiplier to another multiplier at a specified frequency. Previous IQC's for time-delay such as the ones in [3] were shown to correspond to special cases arising from particular choices of these multipliers. This means that the results of [5] and [8] generally produce tightest, least conservative IQC robustness bounds for systems with uncertain time delays.

But, there is an important difficulty with the results of [5] and [8]. Switching from one frequency-depending multiplier to a constant multiplier results in an irrational multiplier, which means that the standard KYP cannot be applied. Frequency sweeping could be used to bypass the KYP lemma, but no matter how fine the frequency grid, there always remains a small risk that a crucial frequency will be missed, resulting an erroneous prediction of robustness. In this paper, we show how to solve this problem by employing the recent *finite frequency strictly positive real lemma* of T. Iwasaki *et al.* [9]. Our main result is a finite-dimensional LMI representation of switched rational-multiplier IQC's. The result eliminates the need for, and the risks of, frequency-sweeping in testing the delay-dependent IQC robustness conditions of [5] and [8].

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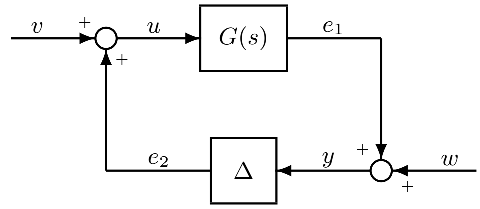

Fig. 1. Basic feedback configuration

The paper is organized as follows: Preliminary background is covered in Section II and the problem formulation is in Section III. Our main result is given in Section IV. Finally conclusions are in Section V.

II. Preliminaries

This section briefly covers preliminary results such as multiplier IQC's for time-delay by M. Jun *et al.* [8] and finite frequency positive real condition by T. Iwasaki *et al.* [9]. Notation used in the paper is standard. $\mathbb{R}$ ($\mathbb{R}_+$) denotes the set of all (positive) real numbers and $\mathbb{C}$ denotes the set of all complex numbers. $A(s)^*$ means para-Hermitian conjugate, that is, $A(-s)^T$. A^{-*} is abbreviation of $(A^*)^{-1}$. I_q denotes $q \times q$ identity matrix. herm(m) and skew(m) are Hermitian and skew part of m, that is, $\frac{1}{2}(m+m^*)$ and $\frac{1}{2}(m-m^*)$, respectively. $\mathfrak{Re}(\cdot)$ ($\mathfrak{Im}(\cdot)$) denotes the real (imaginary) part of $(\cdot)$. $\hat{x}(j\omega)$ means Fourier transform of the signal $x(t)$.

We consider the feedback system in Figure 1 where G and Δ are bounded causal operators on $\mathcal{L}_{2e}^m[0,\infty)$ and $\mathcal{L}_{2e}^l[0,\infty)$, respectively. The multiplier IQC's for time-delay uncertainty by M. Jun *et al.* [8] is stated as follows:

Theorem 1: [8] Let $\Delta(j\omega) \triangleq \delta(j\omega)I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \le \tau \le \bar{\tau}$. Then, $\Delta(j\omega)$ satisfies the IQC defined by

$$\Pi_M(j\omega) \triangleq \begin{cases} \mathrm{herm}\left(\begin{bmatrix} I_q \\ -I_q \end{bmatrix} M(j\omega) \begin{bmatrix} -e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q \end{bmatrix}\right), & |\omega| < \frac{2\pi}{\bar{\tau}} \\ \mathrm{herm}\left(\begin{bmatrix} M(j\omega) & 0 \\ 0 & -M(j\omega) \end{bmatrix}\right), & |\omega| \ge \frac{2\pi}{\bar{\tau}} \end{cases} \quad (1)$$

where $M(j\omega) \in \mathbb{C}^{q\times q}$ and herm$(M(j\omega)) \ge 0$ for all $|\omega| < \frac{2\pi}{\bar{\tau}}$. □

The definition for finite frequency strictly positive real transfer function can be stated as follows:

Definition 1: [9] A stable square transfer function $G(s)$ is said to be *finite frequency strictly positive real (FFSPR)* with bandwidth $\bar{\omega}$ if

$$G(j\omega) + G(j\omega)^* > 0, \quad \forall\, |\omega| \le \bar{\omega}. \quad (2)$$

□

The following lemma is a generalization of the KYP Lemma where a frequency domain condition is required to hold only for a given finite frequency range.

Lemma 1 (Finite-Frequency KYP [9]) Let a real scalar $\omega_0 > 0$, matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ and a Hermitian matrix $\Pi \in \mathbb{R}^{(n+m)\times(n+m)}$ be given. Suppose A has no eigenvalues on the imaginary axis. Then the following statements are equivalent:

i) For all ω such that $|\omega| \leq \omega_0$,

$$\begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix}^* \Pi \begin{bmatrix}(j\omega I - A)^{-1}B \\ I\end{bmatrix} < 0. \tag{3}$$

ii) There exist Hermitian matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{n\times n}$ such that $Q > 0$ and

$$\begin{bmatrix}A & B \\ I & 0\end{bmatrix}^* \begin{bmatrix}-Q & P \\ P & \omega_0^2 Q\end{bmatrix} \begin{bmatrix}A & B \\ I & 0\end{bmatrix} + \Pi < 0. \tag{4}$$

□

III. Problem Formulation

First, define a class of multipliers $\mathcal{M}_r^{(n)}$ to be the set of fixed order LTI transfer matrices which satisfy the followings:

Definition 2: Given q and ω_0, an LTI transfer matrix $M(s)$ is said to belong to the class $\mathcal{M}_r^{(n)}(q, \omega_0)$ if $M(s)$ satisfies [1]

$$M(s) = \sum_{j=0}^{n} A_j s^j, \quad A_j \in \mathbb{R}^{q\times q}, \ j = 1, \cdots n, \tag{5}$$

$$M(j\omega) + M(j\omega)^* > 0, \quad \forall\, |\omega| < \omega_0. \tag{6}$$

□

It can be noticed that each multiplier $M(s) \in \mathcal{M}_r^{(n)}$ can be decomposed into a frequency dependent transfer matrix part and a frequency independent parameter part, viz., $M(s) = \Psi(s)^* H \Psi(s)$. For example, if $M(s) = A_1 s + A_0$ where $A_1, A_0 \in \mathbb{R}^{q\times q}$, then $M(s)$ can be decomposed by

$$M(s) = \begin{bmatrix}-sI_q & I_q\end{bmatrix} \begin{bmatrix}0 & 0 \\ A_1 & A_0\end{bmatrix} \begin{bmatrix}sI_q \\ I_q\end{bmatrix} = \Psi(s)^* H \Psi(s).$$

By restricting the class of multiplier $M(j\omega)$ in Theorem 1 to $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$, a rational approximation of $\Pi_M(j\omega)$ for $|\omega| \geq 2\pi/\bar{\tau}$ is readily obtained. Then, problems to be solved are to find a rational approximation of $\Pi_M(j\omega)$ for $|\omega| < 2\pi/\bar{\tau}$ and to find a finite dimensional LMI stability condition. They can be stated as follows:

Problem 1: Find a rational approximation of the term

$$\text{herm}\left(\begin{bmatrix}I_q \\ -I_q\end{bmatrix} M_r(j\omega) \begin{bmatrix}-e^{-j\frac{\bar{\tau}\omega}{2}} I_q & e^{j\frac{\bar{\tau}\omega}{2}} I_q\end{bmatrix}\right) \tag{7}$$

such that IQC (1) remains satisfied by $\Delta(j\omega) = e^{-j\omega\tau} I_q$ for $|\omega| < 2\pi/\bar{\tau}$ and $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$. □

Problem 2: With rational versions of multiplier IQC in Theorem 1 on two separate frequency range, find a finite dimensional LMI condition for stability of a given time-delay system. □

IV. Main Results

A. Rational Multiplier IQC

Let $\begin{bmatrix}\tilde{y}(j\omega) \\ \tilde{u}(j\omega)\end{bmatrix} \triangleq S_r(j\omega) \begin{bmatrix}y(j\omega) \\ u(j\omega)\end{bmatrix}$ where $S_r(j\omega)$ is invertible, let $u(j\omega) = \Delta(j\omega) y(j\omega)$, and let $\tilde{u}(j\omega) = \widetilde{\Delta}(j\omega)\tilde{y}(j\omega)$. $S_r(j\omega)$ is called *sector transformation matrix* (See [11]). The non-rationality of Eq. (7) is due to the exponential function that was originated from non-rational sector transformation matrix.

[1]We may w.l.o.g. confine our attention to real *polynomial* multipliers [10]. When it is needed to work with proper or strictly proper transfer function $M(s)$, one may always substitute for $M(s)$ the equivalent proper or strictly proper multiplier

$$\tilde{M}(s) = \frac{1}{d(s)d(s)^*} M(s)$$

where $d(s)$ is *any* scalar polynomial of degree at least half that of $M(s)$.

So, our first step is to find a rational sector transformation matrix $S_r(j\omega)$ that can lead to the solution of Problem 1. The following lemma proposes a generalization of the sector transformation matrix in [8] and leads to find a rational version of $\Pi_M(j\omega)$.

Lemma 2: Let $p(j\omega) = p^*(-j\omega), \ \ \forall\ \omega$ and let the transformation matrix $S_r(j\omega) \in \mathbb{C}^{2\times 2}$ be

$$\begin{aligned} S_r(j\omega) &= \begin{bmatrix}S_{11}(j\omega) & S_{12}(j\omega) \\ S_{21}(j\omega) & S_{22}(j\omega)\end{bmatrix} \\ &= \begin{bmatrix}p(j\omega) & -p^*(j\omega) \\ -1 & 1\end{bmatrix}, \qquad 0 < \omega < \pi \end{aligned} \tag{8}$$

where $p(j\omega)$ is a scalar complex-valued function such that $-\pi < \measuredangle p(j\omega) \leq -\omega$ for $0 < \omega < \pi$. If $\Delta(j\omega) \in \{\Delta \mid \Delta = e^{-j\varphi}\}$ where $\varphi \in (0, 2\omega)$ when $0 < \omega < \pi$, then $\widetilde{\Delta}(j\omega) \in \mathbb{R}_+ \cup \{0\}$.

Proof:

$$\begin{aligned} \widetilde{\Delta}(j\omega) &= \frac{\tilde{u}(j\omega)}{\tilde{y}(j\omega)} = \frac{S_{21}(j\omega)y(j\omega) + S_{22}(j\omega)u(j\omega)}{S_{11}(j\omega)y(j\omega) + S_{12}(j\omega)u(j\omega)} \\ &= -\frac{1}{\mathfrak{Im}(p(j\omega))} \cdot \frac{1}{\dfrac{\mathfrak{Re}(p(j\omega))}{\mathfrak{Im}(p(j\omega))} + \dfrac{j + j\Delta(j\omega)}{1 - \Delta(j\omega)}} \end{aligned}$$

since $\Delta(j\omega) \neq 1$ and $\mathfrak{Im}(p(j\omega)) \neq 0$. The fact $\Delta(j\omega) = e^{-j\varphi} = \cos\varphi - j\sin\varphi$ leads to

$$\widetilde{\Delta}(j\omega) = -\frac{1}{\mathfrak{Im}(p(j\omega))} \cdot \frac{1}{\cot(\measuredangle p(j\omega_*)) + \cot(\varphi/2)} \geq 0 \tag{9}$$

since $\cot(\measuredangle p(j\omega_*)) + \cot(\varphi/2) \geq 0$ and $\mathfrak{Im}(p(j\omega)) < 0$. This comes from the fact that $\cot(\cdot)$ is strictly decreasing and $-\pi < \measuredangle p(j\omega) \leq -\omega < -\varphi/2 < 0$ when $0 < \omega < \pi$. ■

Comparing with the $S(j\omega)$ in [8], it can be noticed that $S(j\omega)$ in [8] is the case when $p(j\bar{\tau}\omega/2) = e^{-j\bar{\tau}\omega/2}$ and $e^{-j\bar{\tau}\omega/2}$ satisfies all requirements of $p(j\omega)$ in Lemma 2. By Lemma 2, we can find a rational sector transformation $S_r(j\omega)$ and a rational version $\Pi_{M_r}(j\omega)$ for time-delay uncertainty with the rational transformation matrix $S_r(j\omega)$ if we restrict the scope of the function $p(s)$ to be a real rational function.

Theorem 2 (Main Theorem 1) Let $\Delta(j\omega) \triangleq \delta(j\omega) I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \leq \tau \leq \bar{\tau}$. Suppose that $p(j\omega)$ is a scalar real rational function such that

i) $\measuredangle p(j0) = 0$ and $\measuredangle p(j\pi) = -\pi$,

ii) $-\pi < \measuredangle p(j\omega) \leq -\omega, \ \ 0 < \omega < \pi$.

Then, $\Delta(j\omega)$ satisfies the IQC defined by

$$\Pi_{M_r} \triangleq \begin{cases} \text{herm}\left(\begin{bmatrix}-I_q \\ I_q\end{bmatrix} M_r(j\omega) \begin{bmatrix}p(j\frac{\bar{\tau}\omega}{2}) I_q & -p(j\frac{\bar{\tau}\omega}{2})^* I_q\end{bmatrix}\right), & |\omega| < \frac{2\pi}{\bar{\tau}} \\ \text{herm}\left(\begin{bmatrix}M_r(j\omega) & 0 \\ 0 & -M_r(j\omega)\end{bmatrix}\right), & |\omega| \geq \frac{2\pi}{\bar{\tau}} \end{cases} \tag{10}$$

where $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$.

Proof: For simplicity, only the case when $\omega \geq 0$ is considered. It is proved similarly when $\omega < 0$.

It is trivial to check when $\omega = 0$. Next, when $0 < \omega < \frac{2\pi}{\bar{\tau}}$, it can be noticed that

$$\begin{aligned} &2 \cdot \text{herm}\left(\begin{bmatrix}-I_q \\ I_q\end{bmatrix} M_r(j\omega) \begin{bmatrix}p(j\frac{\bar{\tau}\omega}{2}) I_q & -p(-j\frac{\bar{\tau}\omega}{2}) I_q\end{bmatrix}\right) \\ &= S_r(j\omega)^* \begin{bmatrix}0 & M_r(j\omega)^* \\ M_r(j\omega) & 0\end{bmatrix} S_r(j\omega) \end{aligned}$$

where $S_r(j\omega)$ is defined by $S_r(j\omega) \triangleq \begin{bmatrix} p(j\bar{\tau}\omega/2)I_q & -p(-j\bar{\tau}\omega/2)I_q \\ -I_q & I_q \end{bmatrix}$. Also it can be said by Eq. (9) in Lemma 2 that

$$\begin{aligned}
&\begin{bmatrix} I_q \\ \widetilde{\Delta}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} \begin{bmatrix} I_q \\ \widetilde{\Delta}(j\omega) \end{bmatrix} \\
&= \widetilde{\Delta}(j\omega)M_r(j\omega) + M_r(j\omega)^*\widetilde{\Delta}(j\omega) \\
&= 2\widetilde{\Delta}(j\omega)\,\mathrm{herm}(M_r(j\omega)) \geq 0
\end{aligned}$$

since $\mathrm{herm}(M_r(j\omega)) \geq 0$ by assumption. This leads to

$$\begin{aligned}
0 &\leq \begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} \begin{bmatrix} \tilde{y}(j\omega) \\ \tilde{u}(j\omega) \end{bmatrix} \\
&= \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} y(j\omega) \\ u(j\omega) \end{bmatrix}.
\end{aligned}$$

Now, let us consider the case when $\omega \geq \frac{2\pi}{\bar{\tau}}$. We have

$$\begin{aligned}
&\begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} I_q \\ \Delta(j\omega) \end{bmatrix} \\
&= \mathrm{herm}(M_r(j\omega))\big(I_q - \Delta(j\omega)^*\Delta(j\omega)\big) = 0.
\end{aligned}$$

■

The next problem is how to find a transfer function $p(s)$ which satisfies the given phase requirements. There are myriad of rational or polynomial transfer function $p(s)$ such that the conditions i) and ii) in Theorem 2 are satisfied. Here, we propose an easy method to find such a $p(s)$ using a rational approximation of e^{-s}. We will use the following fact:

Proposition 1: Let $f(s) \triangleq (1-\frac{s}{2n})^n/(1+\frac{s}{2n})^n$, the n^{th} order rational approximation of e^{-s}. Then, $-\omega \leq \measuredangle f(j\omega) \leq 0$ for $0 \leq \omega \leq \pi$ and $\measuredangle f(j0) = 0$. Furthermore, $g(\omega) \triangleq \omega + \measuredangle f(j\omega)$ is monotonically increasing function of ω.

Proof: See appendix. ■

The Proposition 1 says that the phase of $f(s)$ always leads the phase of e^{-s}. Therefore, the main idea in getting an appropriate $p(s)$ is to put more delay to $f(s)$, viz., define $p(s)$ to be $p(s) = (1-\frac{\alpha s}{2n})^n/(1+\frac{\alpha s}{2n})^n$. The value of α is determined so that the phase of $p(j\pi)$ is same as that of $e^{-j\pi}$. Then, the next theorem stipulates that such $p(s)$ satisfies the requirements i) and ii) in Theorem 2.

Proposition 2: Let $p(s) \triangleq (1-\frac{\alpha s}{2n})^n/(1+\frac{\alpha s}{2n})^n$ where α is defined as $\alpha \triangleq (2n/\pi)\tan(\pi/(2n))$. Then, $p(s)$ satisfies the requirements i) and ii) in Theorem 2.

Proof: See appendix. ■

B. Finite Frequency Passivity and Non-Frequency Dependent LMI Formulation

A sufficient stability condition of a given time-delay system can be derived by applying IQC Theorem in conjunction with $\Pi_{M_r}(j\omega)$ in Theorem 1. It is, however, not possible to reliably check the condition

$$\begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix} \leq -\tilde{\epsilon}I, \quad \forall\,\omega \in \mathbb{R}, \qquad (11)$$

by direct frequency sweeping, since this requires a finite gridding of the frequency continuum $\omega \in [0,\infty]$. Therefore it is necessary to convert frequency dependent LMI condition (11) to non-frequency dependent LMI condition. In this section, we provide a non-frequency dependent finite dimensional LMI condition for stability of time-delay system by using the results in the previous section and by applying Lemma 1.

First, define some variables needed for our LMI formulation. As mentioned in Section III, the multiplier $M_r(j\omega) \in \mathcal{M}_r^{(n)}$ can be decomposed into $M_r(j\omega) = \Psi(j\omega)^* H \Psi(j\omega)$ where $\mathrm{herm}(H) > 0$. Define A_1, B_1, C_1 and D_1 to be

$$\begin{bmatrix} \Psi(s) & 0 \\ 0 & \Psi(s) \end{bmatrix} S_r(s) \begin{bmatrix} G(s) \\ I \end{bmatrix} \triangleq C_1(sI - A_1)^{-1}B_1 + D_1. \qquad (12)$$

Let $\hat{G}(s) \triangleq G(1/s)$ and assume that there exists a state-space realization of $\hat{G}(s)$. Define A_2, B_2, C_2 and D_2 to be

$$\begin{bmatrix} \hat{G}(s) \\ I \end{bmatrix} \triangleq C_2(sI - A_2)^{-1}B_2 + D_2. \qquad (13)$$

Then the frequency dependent LMI condition for stability (11) can be transformed to non-frequency dependent LMI condition with rational approximation.

Theorem 3 (Main Theorem 2) For a given $\bar{\tau} > 0$ suppose that the time-delay uncertainty is expressed as $\Delta(j\omega) \triangleq \delta(j\omega)I_q$ where $\delta(j\omega) = e^{-j\omega\tau}$ and $0 \leq \tau \leq \bar{\tau}$. Let $\dot{w} \in \mathcal{L}_{2e}^q[0,\infty)$. Assume that:

i) the feedback system is finite-gain stable when $\tau = 0$,
ii) $G(s)$ is strictly proper,
iii) $\hat{G}(s) = G(1/s)$ is proper,
iv) there exists an Q_1, Q_2, H and M such that

$$\begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D_1 \end{bmatrix}^* \begin{bmatrix} -Q_1 & P_1 & 0 \\ P_1 & \left(\frac{2\pi}{\bar{\tau}}\right)^2 Q_1 & 0 \\ 0 & 0 & \begin{bmatrix} 0 & H^* \\ H & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} A_1 & B_1 \\ I & 0 \\ C_1 & D_1 \end{bmatrix} < 0, \qquad (14)$$

$$\begin{bmatrix} A_2 & B_2 \\ I & 0 \\ C_2 & D_2 \end{bmatrix}^* \begin{bmatrix} -Q_2 & P_2 & 0 \\ P_2 & \left(\frac{\bar{\tau}}{2\pi}\right)^2 Q_2 & 0 \\ 0 & 0 & \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ I & 0 \\ C_2 & D_2 \end{bmatrix} < 0, \qquad (15)$$

$$Q_1 > 0, \;\; Q_2 > 0, \;\; H + H^* > 0, \;\; M = M^* > 0. \qquad (16)$$

where A_i, B_i, C_i and D_i, $i = 1,2$ are defined as in Eq. (12) and Eq. (13).

Then, the feedback system in Figure 1 is robust against the time-delay $0 \leq \tau \leq \bar{\tau}$ in the sense that the mapping $(v, w, \dot{w}) \mapsto (u, y)$ is finite-gain stable.

Proof: By IQC Theorem, we just have to show that the statement

There exists an $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$ such that for some $\tilde{\epsilon} > 0$

$$\begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix}^* \Pi_{M_r}(j\omega) \begin{bmatrix} G(j\omega) \\ I_q \end{bmatrix} \leq -\tilde{\epsilon}I, \quad \forall\,\omega \in \mathbb{R},$$

is equivalent to the condition iv).

First, consider the stability condition when $|\omega| < 2\pi/\bar{\tau}$. By Theorem 2 and Eq. (11), the stability condition on this frequency range can be stated as

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* S_r(j\omega)^* \begin{bmatrix} 0 & M_r(j\omega)^* \\ M_r(j\omega) & 0 \end{bmatrix} S_r(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0 \qquad (17)$$

for $M_r(j\omega) \in \mathcal{M}_r^{(n)}(q, 2\pi/\bar{\tau})$. Since $M(j\omega)$ can be decomposed into $M(j\omega) = \Psi(j\omega)^* H \Psi(j\omega)$, the Eq. (17) is equivalent to saying that

$$\begin{aligned}
&\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* S_r(j\omega)^* \begin{bmatrix} \Psi(j\omega) & 0 \\ 0 & \Psi(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & H^* \\ H & 0 \end{bmatrix} \cdot \\
&\qquad \begin{bmatrix} \Psi(j\omega) & 0 \\ 0 & \Psi(j\omega) \end{bmatrix} S_r(j\omega) \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \qquad (18)
\end{aligned}$$

$$H + H^* > 0. \tag{19}$$

Therefore, by Lemma 1, the frequency dependent LMI (18) and (19) are equivalent to the frequency independent LMI conditions Eq. (14), $Q_1 > 0$, and $H + H^* > 0$.

Next, let us consider stability condition for $|\omega| \geq 2\pi/\bar{\tau}$. We can notice that the condition

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} < 0, \quad |\omega| \geq \frac{2\pi}{\bar{\tau}}$$

is equivalent to the condition

$$\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} < 0, \quad |\omega| \leq \frac{\bar{\tau}}{2\pi}. \tag{20}$$

Thus, by Lemma 1, the condition (20) is equivalent to frequency independent LMI conditions Eq. (15), $Q_2 > 0$, and $M = M^* > 0$. ■

The properness of $\hat{G}(s)$ is not restrictive much. In practice, an additional parameter $\epsilon \ll 1$ can be added in order to make $\hat{G}(s)$ proper when it is not proper. For example, when $G(s) = 1/s$, then approximate $\hat{G}(s) = s/(\epsilon s + 1)$.

When synthesis problem is considered, a controller $K(s) = C_K(sI - A_K)^{-1}B_K + D_K$ is inserted into the general feedback configuration in Figure 1 and synthesis problem becomes similar to μ synthesis problem. This is BMI problem [12], not LMI problem, since the LMI's in Theorem 3 is affine in M, H, Q_1, Q_2 and controller parameters(A_K, B_K, C_K, D_K) respectively but not jointly. Still, we may use suboptimal *D-K iteration-like* method, viz., first set initial controller parameters A_K, B_K, C_K and D_K, and then solve LMI's in Theorem 3 with variables M, H, Q_1 and Q_2. If LMI's are not feasible, fix the variables M, H, Q_1, Q_2 with the values which yield minimum and then solve the LMI's with variables A_K, B_K, C_K and D_K. Repeat this iteration until the LMI's are feasible with desired time-delay $\bar{\tau}$ is obtained.

Example 1: Consider the following autonomous system

$$\dot{x}(t) = A_0 x(t) + A_d x(t - \tau), \qquad 0 \leq \tau \leq \bar{\tau}$$

with $A_0 = \left[\begin{smallmatrix} -2 & 0 \\ 0 & 0.25 \end{smallmatrix}\right]$ and $A_d = \left[\begin{smallmatrix} -1 & 0 \\ -0.1 & -0.85 \end{smallmatrix}\right]$ which is the same example considered in Fu *et al.* [7]. The second order transfer function $p(s) = \frac{0.1013s^2 - 0.6366s + 1}{0.1013s^2 + 0.6366s + 1}$ is used, which can be obtained by Proposition 2. The estimate of maximum delay margin using Theorem 1 is $\tau_{max} = 1.489$.

As comparisons, the estimate of allowable maximum time delay is $\tau_{max} = 0.6417$ when the Theorem 6 in [7] is applied and it is $\tau_{max} = 0.9848$ when the Theorem 7 in [7] is applied. It is $\tau_{max} = 0.9999$ with application of LMI condition by [6] while the optimal value for the system with the given parameters, A_0 and A_d, is $\tau_{opt} = 1.54$ [7]. It can be said that the result of this paper is less conservative than the one in [7] and [6], and very close to the optimal value. □

V. Conclusion

In an effort to find a rational version of $\Pi_M(j\omega)$ in [8], a generalized sector transformation matrix $S_r(j\omega)$ which transforms an arc to positive real ray or line segment in complex plane was provided. The $S_r(j\omega)$ is parameterized in terms of scalar complex-valued function $p(j\omega)$ which satisfies a certain phase conditions. An easy algorithm to find one of such $p(j\omega)$'s using a rational approximation of e^{-s} was proposed, and a class of rational $\Pi(j\omega)$'s which define the IQC satisfied by time-delay uncertainty which are linearly parameterized in terms of a real rational multiplier $M_r(j\omega)$ was also provided. Finally, a set of non-frequency dependent LMI's to check the robust stability of time-delay systems with application of finite frequency positive real condition by [9] was presented, which opens the way to delay-dependent robust controller synthesis via *D-K* iteration-like techniques.

Appendix

Proof of Proposition 1: $\measuredangle f(j\omega) = -2n \cdot \arctan(\omega/(2n))$ since $f(s) = (1 - \frac{s}{2n})^n/(1 + \frac{s}{2n})^n$. We get $\measuredangle f(j\omega) \leq 0$ since $\arctan(\omega/(2n)) \geq 0$ for $0 \leq \omega \leq \pi$. The fact that $g(w)$ is monotonically increasing function of ω for each n is proved by $\frac{dg(\omega)}{d\omega} = 1 - 1/\left(1 + (\frac{\omega}{2n})^2\right) \geq 0$ and $-\omega \leq \measuredangle f(j\omega)$ comes from $g(\omega) = \omega + \measuredangle f(j\omega) \geq 0$. ■

Proof of Proposition 2: For simplicity, we consider only the case when $0 < \omega < \pi$. The case when frequency is negative can be proved in the same way. Let $h(\omega) \stackrel{\triangle}{=} \measuredangle p(j\omega)$, that is, $h(\omega) \stackrel{\triangle}{=} -2n \cdot \arctan(\alpha\omega/(2n))$. Then, it is easily checked that $h(0) = 0$ and $h(\pi) = -\pi$. Since $\frac{dh(\omega)}{d\omega} < 0$, $h(\omega)$ is strictly decreasing function of ω, thus, $h(\omega) > h(\pi) = -\pi$ for $0 < \omega < \pi$. And we get $h(\omega) \leq -\omega$ from the fact that $\left.\frac{d(h(\omega)+\omega)}{d\omega}\right|_{\omega=0} < 0$, $\left.\frac{d(h(\omega)+\omega)}{d\omega}\right|_{\omega=\pi} > 0$ and the function $h(\omega) + \omega$ is convex function of ω. ■

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