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An Algorithm to Compute Multipliers for Repeated Monotone Nonlinearities

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Abstract – A MIMO negative feedback interconnection of a discrete-time LTI system and a repeated time-invariant monotone nonlinearity is considered. Kulkarni and Safonov derived in 2001 the largest class of multipliers that can be used to show stability of such feedback systems. This note demonstrates in a simple way that these multipliers can be computed efficiently from the solution of a Linear matrix inequality (LMI). It is also shown how the structure of the problem can be exploited to reduce its complexity.

I. INTRODUCTION

Zames and Falb [3] studied continuous-time SISO feedback systems of the form in Figure 1, and introduced a broad class of allowable multipliers that can be used to prove finite-gain stability of such systems. Although these stability conditions in terms of multipliers are of great theoretical value, the efficient and reliable computation of such multipliers remains a problem to be addressed.

Various algorithms along with its corresponding geometrical interpretations have been proposed to compute multipliers. For example, in 1987 Safonov and Wyetzner [17] demonstrated that finite-impulse approximations of Zames and Falb multipliers could be computed in the continuous-time SISO case as a large scale linear programming problem. Gapski and Geromel [16] exploited the convexity of the problem to design a numerical method that reduced the large dimensionality of the problem, also in the continuous-time SISO case. Chen and Wen [18] considered the MIMO continuous-time case, with diagonal nonlinearity using rational approximations of Zames-Falb multipliers and LMIs. Kothare and Morari [19] used the Chen and Wen algorithm to study the stability of anti-windup control systems.

Recently, D'Amato et al. [21] obtained, a class of positivity preserving multipliers, for the repeated SISO monotone nonlinearities. It was unknown, if this was the largest class of positivity preserving multipliers for this type of repeated nonlinearities. In 2001 Kulkarni and Safonov [22], relaxed D'Amato's conditions on the multiplier, and characterized the largest class of multipliers that preserve positivity of such nonlinearity, thus reducing conservativeness of stability analysis for this type of feedback systems.

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In this note the numerical computation of the multipliers derived by Kulkarni and safonov [22] is adressed. A discrete-time MIMO feedback system of the form in Figure 1 is considered. It is demonstrated in a simple way that these multipliers can be obtained from the solution of an LMI. The structure of the problem is exploited to reduce the complexity of the numerical problem.

The results are organized as follows. In section 2, necessary terminology is introduced and background results are stated in section 3. The problem of concern is formally posed in section 4. Main results are presented in section 5. Finally some conclusions are drawn in section 6.

II. PRELIMINARIES

The notation used is summarized in Table 1.

Table 1: Notation

Symbol	meaning
$\Re$	set of real numbers
l_2	space of infinite sequences $z = \{z(k)\}$ such that $\sum \lvert z(k)\rvert^2$ is summable
l_{2e}	extended l_2 space, that is the space of infinite sequences $z = \{z(k)\}$ such that $z_T \in l_2$, where $z_T(k) = z(k)$ if $k \leq T$, $z_T(k) = 0$ if $k > T$
l_1	space of sequences $\{z(k)\}$ such that $\sum \lvert z(k)\rvert$ is summable
$\lVert x(k)\rVert_2$	$\left(\sum \lvert x(k)\rvert^2\right)^{\frac{1}{2}}$, where $x(k) \in \Re^n$ and $\lvert . \rvert$ is any norm on $\Re^n$
I_n	$n \times n$ identity matrix
herm(D)	$\frac{1}{2}(D + \overline{D}')$ for $D \in C^{n\times n}$ or $\Re^{n\times n}$
$\underline{U} > 0$	positive semidefinite matrix U
$\overline{A}$	the complex conjugate of the matrix A
$A\prime$	the transpose of A
A^*	$\overline{A}\prime$ for a matrix A
MIMO	multi-input-multi-output
SISO	single-input-single-output
LMI	linear matrix inequality
LTI	linear time invariant system

A few definitions are in order before we can formally pose our problem.

Definition 1 [l_2 Stability, [6]]

A causal operator $h : l_{2e} \to l_{2e}$ is said to be l_2 stable if $x \in l_2$ implies $hx \in l_2$. Furthermore, if $\exists\ \gamma \geq 0$ and β such that $\|hx\|_2 \leq \gamma \|x\|_2 + \beta \quad \forall x \in l_2$, then, h is said to be finite-gain l_2 stable.

The feedback system shown in Figure 1, is (finite-gain) l_2 stable if all closed loop maps from all external inputs to all internal variables are (finite-gain) l_2 stable.

Remark 1

It is straightforward to check that it is sufficient to find $\gamma < \infty$, β such that

$$\|e_i\|_2 \leq \gamma(\|x_1\|_2 + \|x_2\|_2) + \beta \quad i = 1, 2 \tag{1}$$

to show that the feedback system in Figure 1 is finite-gain l_2 stable.

Definition 2 [monotone nonlinearity]

A nonlinearity $\mathsf{N_s} : \Re \to \Re$ is said to belong to the set N_s of SISO memoryless monotone nonlinearities iff $\mathsf{N_s}$ satisfies the conditions:

a) $\mathsf{N_s}$ is monotone nondecreasing, i.e., $(r - t)(\mathsf{N_s}(r) - \mathsf{N_s}(t)) \geq 0$;

b) There is a constant $c > 0$ such that $|\mathsf{N_s}(r)| \leq c\,|r|$ for all $r \in \Re$.

Remark 2

Definition 2 is as given in Zames and Falb [3].

Definition 3 [repeated monotone nonlinearity]

The set N_{rs} of repeated SISO monotone nonlinearities consists of elements $\mathsf{N_{rs}} : \Re^n \to \Re^n$ such that for $p \in \Re^n$:

$$N_{rs}(p) = col\{N_s(p_1), ..., N_s(p_n)\}, N_s \in N_s \tag{2}$$

Definition 4 [set M_{rs} of multipliers for N_{rs}]

M_{rs} denotes the set of discrete MIMO convolution operators, such that for $\mathsf{M} \in \mathsf{M_{rs}}$ we have:

a) $\mathsf{M} \in \mathsf{L}(\mathsf{l_2}, \mathsf{l_2})$.

b) The impulse response $m(n)$ of M satisfy the following 'row' condition : for $i = 1, ..., n$

$$m_{ii}(0) \geq \sum_{k=-\infty, k\neq 0}^{\infty} |m_{ii}(k)| + \sum_{k=-\infty}^{\infty} \sum_{j=1, j\neq i}^{n} |m_{ij}(k)| \tag{3}$$

and the 'column' condition: for $j = 1, ..., n$

$$m_{jj}(0) \geq \sum_{k=-\infty, k\neq 0}^{\infty} |m_{jj}(k)| + \sum_{k=-\infty}^{\infty} \sum_{i=1, i\neq j}^{n} |m_{ij}(k)| \tag{4}$$

c) And the 'nonpositivity' condition

$$m_{ss}(0) = 1 \text{ for } s \in \{1, \ldots, n\} \text{ and } m_{ij}(k) \leq 0,\ i \neq j \tag{5}$$

Remark 3

Definitions 3 and 4 were introduced in Kulkarni and Safonov [22]. Definition 4 says that M is a convolution operator, whose associated array $\{m_{kl}\}$ is doubly hyperdominant in the sense of Willems [5].

III. BACKGROUND RESULTS

The following well known results are used:

Lemma 1 [l_2 finite-gain Stability, [22]]

Consider the discrete time MIMO feedback system in Figure 1, where T is an LTI causal finite-gain stable system and N a nonlinearity in N_{rs}. If there exists a multiplier $\mathsf{M} \in \mathsf{M_{rs}}$ such that:

$$\text{herm}\left\{\mathsf{M}(e^{j\omega})\mathsf{T}(e^{j\omega})\right\} \geq 0 \text{ for } \omega \in [0, 2\pi] \tag{6}$$

then the feedback system is l_2 finite-gain stable.

Remark 4

Sufficient conditions for finite-gain stability of a SISO system of the form as in Figure 1 were introduced by Zames and Falb [3], using the theory of positive operators on a Hilbert space. For the MIMO continuous time case and the nonlinearity N in N_{rs} D'Amato et al. [21] obtained, a larger class of positivity preserving multipliers. Finally, Kulkarni and Safonov [22] characterized the largest class of positivity preserving multipliers M_{rs} by relaxing some conditions on D'amato's multipliers.

Lemma 2 [Positive Real lemma, [4]]

A matrix transfer function $\mathsf{H(s)}$ having no poles on the $j\omega$ axis, with a controllable state-space realization (A, B, C, D), satisfies

$$\text{herm}\{\mathsf{H(s)}\} \geq 2\varepsilon \mathsf{I}\ ,\quad s = j\omega \quad \forall \omega$$

iff there exists a symmetric matrix $P = P'$, $\det(P) \neq 0$ such that

$$\text{herm}\left\{\begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ -C & -D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon I \end{bmatrix}\right\} \leq 0 \tag{7}$$

Remark 5

The original version of lemma 2 in [4] is stated in a different but equivalent form. Here it is presented in such a way that the linear dependence in P is clearly observed.

IV. PROBLEM FORMULATION

Consider the MIMO discrete time feedback system illustrated in Figure 1 under the following assumptions.

A1. $x_i, y_i, e_i,\ i = 1, 2$ are n-dimensional discrete real signals.

A2. $\mathsf{T} = [\mathsf{T}_{ij}]_{n\times n}$, is a discrete LTI causal and stable operator.

A3. N is an element of N_{rs}.

Problem 1 [Multiplier Problem (MP)]

For the feedback system in Figure 1, and under the assumptions A1-A3, compute a multiplier $\mathsf{M} \in \mathsf{M}_{rs}$ such that

$$\text{herm}\left\{\mathsf{M}(e^{j\omega})\mathsf{T}(e^{j\omega})\right\} \geq 0 \quad \text{for } \omega \in [0, 2\pi]$$

or show that there is no such multiplier.

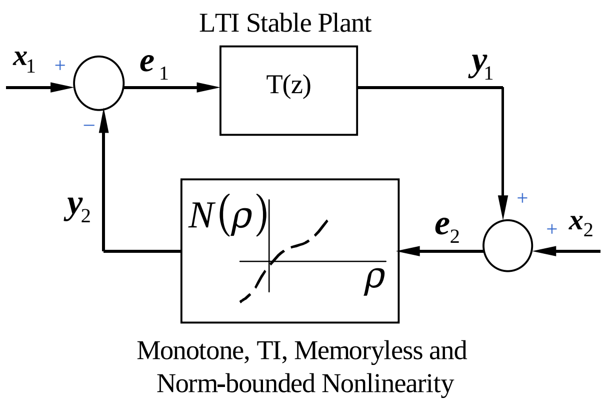

Fig. 1. MIMO feedback system

V. MAIN RESULTS

Let us approximate M as follows:

$$\mathsf{M} = \sum_{k=-N}^{N} m(k) z^{-k}$$

where $z = e^{j\theta}$, $N \geq 0$ is assumed to be fixed for the moment, and $m(k)$ are $n\times n$ matrices to be determined.

Lemma 4

The frequency-domain condition $\text{herm}\{\mathsf{M}\,\mathsf{T}\} \geq 0$ can be stated as an LMI.

Proof:

Before we are able to apply the continuous time version of the Positive Real lemma, we require some preliminary steps:

$$M(z)T(z) = [m(-N)z^N + ... + m(N)z^{-N}]T(z) =$$

$$[m(-N)\cdots m(0)\cdots m(N)] \begin{bmatrix} z^N I_n \\ \vdots \\ I_n \\ \vdots \\ z^{-N} I_n \end{bmatrix} T(z) \qquad (8)$$

We factor the noncausal term z^N as $z^N = R_+ R_-$ where R_+, R_-^{-1}, R_-^{*-1} are causal. For example take

$$R_+ = z^N \frac{p(z)}{q(z)} \qquad \text{and} \qquad R_- = \frac{q(z)}{p(z)} \qquad (9)$$

where $p(z)$ and $q(z)$ are polynomials in z such that

$$\text{degree}\{p(z)\} + N \leq \text{degree}\{q(z)\}$$

Also,

$$\text{herm}\{\mathsf{M}\,\mathsf{T}\} \geq 0 \text{ if and only if } \text{herm}\{\mathsf{S}^* \mathsf{M}\,\mathsf{T}\mathsf{S}\} \geq 0$$

for any $n \times n$ invertible matrix S.

Let S be the invertible and diagonal $n \times n$ matrix $\mathsf{S} = R_-^{-1} I_n$. Therefore,

$$\text{herm}\{\mathsf{S}^* \mathsf{M}\,\mathsf{T}\mathsf{S}\} = \text{herm}\left\{\mathsf{R}_-^{-1*}\mathsf{I}_n \mathsf{M}\,\mathsf{T}\mathsf{R}_-^{-1}\mathsf{I}_n\right\} \qquad (11)$$

Define

$$\widetilde{\mathsf{M}} \triangleq [m(-N)\cdots m(0)\cdots m(N)] \qquad (12)$$

$$\mathsf{F}(z) = \begin{bmatrix} R_-^{-1*} R_+ I_n \\ \vdots \\ R_-^{-1*} R_-^{-1} I_n \\ \vdots \\ z^{-N} R_-^{-1*} R_-^{-1} I_n \end{bmatrix} \qquad (13)$$

thus, from (8) − (13) we conclude that

$$\text{herm}\{\mathsf{M}\,\mathsf{T}\} \geq 0 \text{ if and only if } \text{herm}\left\{\widetilde{\mathsf{M}}\,\mathsf{F}\,\mathsf{T}\right\} \geq 0$$

where $\widetilde{\mathsf{M}}$, F, T are proper and rational transfer functions.

We choose a minimal realization of $\mathsf{T}(z)$

$$\mathsf{T}(z) \sim \begin{bmatrix} A_T & B_T \\ C_T & D_T \end{bmatrix}$$

where A_T, B_T, C_T, D_T are known.

We find also a controllable realization for $\mathsf{F}(z)$

$$\mathsf{F}(z) \sim \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix}$$

where A_F, B_F, C_F, D_F are all known, and determined by the choice of R_+, R_- and N.

We apply a bilinear ('Tustin') transformation of the form

$$z = \frac{\frac{t}{2}s + 1}{-\frac{t}{2}s + 1} \triangleq \frac{\alpha s + \delta}{\gamma s + \beta} \quad \text{where } t \text{ is a fixed number.}$$

to find continuous time state space equivalent systems:

$$\mathsf{T}(s) \sim \begin{bmatrix} \widetilde{A}_T & \widetilde{B}_T \\ \widetilde{C}_T & \widetilde{D}_T \end{bmatrix} =$$

$$\left[\begin{array}{cc}(\beta A_T-\delta I)Q_T & (\alpha\beta-\gamma\delta)Q_T B_T \\ C_T Q_T & D_T+\gamma C_T Q_T B_T\end{array}\right]$$

where

$$Q_T = (\alpha I - \gamma A_T)^{-1}$$

similarly

$$F(s) \sim \left[\begin{array}{cc}\widetilde{A}_F & \widetilde{B}_F \\ \widetilde{C}_F & \widetilde{D}_F\end{array}\right] =$$

$$\left[\begin{array}{cc}(\beta A_F-\delta I)Q_F & (\alpha\beta-\gamma\delta)Q_F B_F \\ C_F Q_F & D_F+\gamma C_F Q_F B_F\end{array}\right]$$

where

$$Q_F = (\alpha I - \gamma A_F)^{-1}$$

thus,

$$\mathrm{herm}\left\{\widetilde{M}F(z)T(z)\right\} \geq 0 \quad \text{for} \quad z = e^{j\theta}, \quad \theta \in [0, 2\pi]$$

if and only if

$$\mathrm{herm}\{\widetilde{M}F(s)T(s)\} \geq 0 \quad \text{for} \quad s = j\omega, \quad \text{all } \omega$$

We find a controllable realization of $\widetilde{M}F(s)T(s)$ in terms of the realizations of $\widetilde{M}$, $F(s)$ and $T(s)$ as

$$\widetilde{M}F(s)T(s) \sim \left[\begin{array}{ccc}\widetilde{A}_T & 0 & \widetilde{B}_T \\ \widetilde{B}_F\widetilde{C}_T & \widetilde{A}_F & \widetilde{B}_F\widetilde{D}_T \\ \widetilde{M}\left[\begin{array}{cc}\widetilde{D}_F\widetilde{C}_T & \widetilde{C}_F\end{array}\right] & & \widetilde{M}\left[\widetilde{D}_F\widetilde{D}_T\right]\end{array}\right],$$

$$\left[\begin{array}{cc}\widetilde{\widetilde{A}} & \widetilde{\widetilde{B}} \\ \widetilde{M}\,\widetilde{\widetilde{C}} & \widetilde{M}\,\widetilde{\widetilde{D}}\end{array}\right] \tag{14}$$

Finally, by the Positive Real lemma

$$\mathrm{herm}\left\{\widetilde{M}F(s)T(s)\right\} \geq 2\varepsilon I_n \quad \text{where } s = j\omega, \text{ all } \omega$$

if and only if there exists $P = P^{\theta}$ such that

$$\mathrm{herm}\left\{\left[\begin{array}{cc}P & 0 \\ 0 & I\end{array}\right]\left[\begin{array}{cc}\widetilde{\widetilde{A}} & \widetilde{\widetilde{B}} \\ -\widetilde{M}\,\widetilde{\widetilde{C}} & -\widetilde{M}\,\widetilde{\widetilde{D}}\end{array}\right]\right\} +$$

$$\mathrm{herm}\left\{\left[\begin{array}{cc}0 & 0 \\ 0 & \varepsilon I_n\end{array}\right]\right\} \leq 0 \tag{15}$$

which is an LMI in P, $\widetilde{M}$, ε. Take $\varepsilon = 0$ to get the result. 2

Lemma 5

The condition that $M \in M_{rs}$ can be written as an LMI.

Proof:

Let

$$M(z) = [m(-N)z^N + ... + m(N)z^{-N}] \tag{16}$$

Define for $i = 1, \ldots, n$

$$e_i \triangleq m_{ii}(0) + \sum_{k=-N, k\neq 0}^{N} m_{ii}(k) + \sum_{k=-N}^{N}\sum_{j=1, j\neq i}^{n} m_{ij}(k)$$

the 'row' condition (3) becomes

$$\mathrm{diag}\{e_1, \ldots, e_n\} \geq 0 \tag{17}$$

similarly, define for $j = 1, ..., n$

$$l_j \triangleq m_{jj}(0) + \sum_{k=-N, k\neq 0}^{N} m_{jj}(k) + \sum_{k=-N}^{N}\sum_{i=1, i\neq j}^{n} m_{ij}(k)$$

thus, the 'column' condition (4) results in

$$\mathrm{diag}\{l_1, \ldots, l_n\} \geq 0 \tag{18}$$

also, the nonpositivity condition (5)

$$m_{ss}(0) = 1, \ s \in \{1, \ldots, n\} \quad \text{and} \quad -m_{ij}(k) \geq 0, \quad i \neq j \tag{19}$$

Clearly conditions (17), (18), (19) form an LMI in the coefficients $m_{ij}(k)$. 2

Theorem 1

The Multiplier Problem (MP) can be posed as an LMI.

Proof:

Follows inmediatelly from lemmas 3 and 4. 2

In fact, we can write it in the standard form:

$$\text{minimize} \qquad c^T x$$

$$\text{subject to} \qquad F(x) > 0 \tag{20}$$

where x is the vector made of the variables in P, $\widetilde{M}$ and ε. Also $c^T x = -\varepsilon$, and $F(x)$ is the block diagonal matrix that consists of the LMI's (15) – (19), with the appropriate modifications and changes in sign to get the $>$ inequality in (20).

Remark 6

A much shorter proof of theorem 1 can be presented, but the one shown give us also an algorithm to compute the multiplier.

Algorithm 1

1. Choose $N > 0$, write $M(z)$ as in (16).
2. Choose R_+ and R_-, and compute a controllable realization $[\widetilde{\widetilde{A}}\ \widetilde{\widetilde{B}}\ \widetilde{\widetilde{C}}\ \widetilde{\widetilde{D}}]$ as in (14).
3. Find x_{opt} the optimum to problem (20).
4. If $\varepsilon < 0$ increase N and go back to 1. Else STOP a multiplier $\widetilde{M}$ has been found.

Algorithm 1 although in principle correct, is computationally very intensive for several reasons: It is possible that only a few $m(k)$ appropriately chosen be enough to solve the problem, and we are choosing 'too many' which will increase the size of the problem considerably, on the other hand if we choose N small and increase it slowly, we will need to compute the matrices P and $\widetilde{M}$ very often which is again very expensive computationally. To address this issue we propose the following modification to algorithm 1.

Define the time samples iteratively.

Suppose we have chosen the time samples $\tau_N = \{t_1, \ldots, t_N\}$ we will choose $\tau_{N+1} = \tau_N \cup \{t_{N+1}\}$ as follows:

An 'unconstrained approximation' of the problem in (20), which leads to what is called method of centers, is given by

$$\text{minimize} \quad L_\lambda(x)$$

with

$$L_\lambda(x) = \ln\left\{\det\left[F(x)^{-1}\right]\right\} + \ln\left\{\frac{1}{\lambda - c^T x}\right\}$$

where $\ln\left\{\det\left[F(x)^{-1}\right]\right\}$ is a self-concordant barrier for the feasible domain $\{x | F(x) > 0\}$ and $\ln\left\{\frac{1}{\lambda - c^T x}\right\}$ is a barrier for the nonnegative half-axis.

Let x_{opt} a solution of (20) for the time samples τ_N, then we choose t_{N+1} in such a way that the directional derivative of $L_\lambda(x_{opt}, 0)$ in the direction $h = (0, \ldots, 0, 1)$ is minimized, or at least negative, this will guarantee us that the optimal solution of (20) with the sample times τ_{N+1} will have a substantial decrease in its cost funtion. This derivative can be computed from

$$DL_\lambda(x)[h] = \nabla L_\lambda(x) \cdot h$$

where the components of $\nabla L_\lambda(x)$ are

$$\frac{\partial L_\lambda}{\partial x_i} = -Tr\left\{F(x)^{-1} F_i\right\} + \frac{c_i}{(\lambda - c^T x)}, \quad i = 1, \ldots, m$$

with

$$F(x) = F_0 + \sum_{i=1}^{m} x_i F_i$$

and λ is chosen so that $\lambda < c^T x_{opt}$.

VI. CONCLUSIONS

In this paper an algorithm that for the Multiplier Problem was proposed. The structure of the problem has been exploited to reduce its computationally complexity. Some numerical examples are being implemented and the results will be presented elsewhere.

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