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Deterministic estimation, control, and the ellipsoid algorithm

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Abstract

The ellipsoid algorithm is applied to the solution of deterministic estimation problems. An algorithm, which either finds a point in the intersection of a finite number of ellipsoids and an ellipsoid which contains that intersection, or determines if the volume of that intersection is smaller than a given value, is obtained. When applied to the solution of deterministic estimation problem, this algorithm produces a sequence of decreasing volume ellipsoids which contains the set of unfalsified candidates. The application to unfalsified control, in particular, produces a cutting plane algorithm for adaptive control, which allows a precise characterization of learning in terms of a sequence of recursively computable shrinking balls of controllers.

Keywords: Deterministic estimation; model validation; unfalsified control; adaptive control; cutting plane; ellipsoid algorithm.

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1 Introduction

Deterministic estimation is basically concerned with unknown quantities which lie in a known set of alternatives [1]. These unknown quantities may be parameters of a candidate model in model validation ([2],[3],[4]) or parameters of a candidate controller in unfalsified control ([5],[6],[7],[8],[9],[10],[11],[12],[13],[14]). In many cases, the known set of alternatives or candidates is given by an intersection of ellipsoids as can be seen in [15] for the unknown but bounded approach to recursive state estimation, in [1] for the linear estimation problem with normalized nonparametric uncertainty, and in [10] for the unfalsified model reference adaptive control problem. It is relevant, then, to look for a solution to the problem of finding a point in the intersection of a finite number of ellipsoids or determining if the volume of the intersection is smaller than a given value.

A solution to this problem can be found by applying the ellipsoid algorithm [16]. First, we present a modified version of the ellipsoid algorithm which allow us to find a point in the intersection of a finite number of ellipsoids. Then we present formulations of sets of candidate models/controllers, comment on the practical aspects of the algorithm and apply it to examples corresponding to the formulated sets of candidates. This application generate a recursive cutting plane algorithm for adaptive control. Finally, we present our conclusions and comments.

2 Finding a point in the intersection of a finite number of ellipsoids

2.1 An Ellipsoid Algorithm

A version of the ellipsoid algorithm is given in [16]. We will now give a modified version of this algorithm which allow us to find a point in the intersection of a finite number of ellipsoids. The basic idea of the algorithm is as follows. We start with an ellipsoid $\mathcal{E}^{(0)}$ that is obviously guaranteed to contain the intersection of all the ellipsoids. We then compute a cutting plane that passes through the center $x^{(0)}$ of $\mathcal{E}^{(0)}$. This means that we find a nonzero vector $g^{(0)}$ such that the intersection of all the ellipsoids lies in the half-space $\{z \mid g^{(0)T}(z - x^{(0)}) < 0\}$. We then know that the sliced half-ellipsoid

$$\mathcal{E}^{(0)} \cap \{z \mid g^{(0)T}(z - x^{(0)}) < 0\}$$

contains the intersection of all the ellipsoids. Now we compute the ellipsoid $\mathcal{E}^{(1)}$ of minimum volume that contains this sliced half-ellipsoid; $\mathcal{E}^{(1)}$ is then guaranteed to contain the intersection of all the ellipsoids. The process is then repeated.

2.2 Explicit description of the algorithm

As stated in [16], an ellipsoid $\mathcal{E}$ can be described as

$$\mathcal{E} = \{z \in \mathbb{R}^m \mid (z-a)^T A^{-1}(z-a) \leq 1\}$$

where $A = A^T > 0$. The minimum volume ellipsoid that contains the half-ellipsoid

$$\{z \in \mathbb{R}^m \mid (z-a)^T A^{-1}(z-a) \leq 1, g^T(z-a) \leq 0\}$$

is given by

$$\tilde{\mathcal{E}} = \{z \in \mathbb{R}^m \mid (z-\tilde{a})^T \tilde{A}^{-1}(z-\tilde{a}) \leq 1\},$$

where

$$\tilde{a} = a - \frac{A\tilde{g}}{m+1},$$

$$\tilde{A} = \frac{m^2}{m^2-1}(A - \frac{2}{m+1}A\tilde{g}\tilde{g}^T A), \quad \text{and}$$

$$\tilde{g} = \frac{g}{\sqrt{g^T A g}}.$$

The algorithm is initialized with $k = 0$, $x^{(0)}$ and $\mathcal{E}^{(0)}$ corresponding to one of the ellipsoids. The algorithm then proceeds as follows:

while $x^{(k)}$ does not belong to the intersection

compute a $g^{(k)}$ that defines a cutting plane at $x^{(k)}$

$\tilde{g} := \frac{g^{(k)}}{\sqrt{g^{(k)T} A g^{(k)}}}$

$x^{(k+1)} = x^{(k)} - \frac{A^{(k)}\tilde{g}}{m+1}$

$A^{(k+1)} = \frac{m^2}{m^2-1}(A^{(k)} - \frac{2}{m+1}A^{(k)}\tilde{g}\tilde{g}^T A^{(k)})$

$k := k+1$

Notice that if $x^{(k)}$ does not belong to the ellipsoid

$$\{\theta \mid \theta^T A\theta - 2\theta^T B + C \leq 0\}$$

then a cutting plane at $x^{(k)}$ is given by $g^{(k)} = Ax^{(k)} - B$.

The previous recursion generates a sequence of ellipsoids that are guaranteed to contain the intersection of all the ellipsoids. As stated in [16], it turns out that the volume of these ellipsoids decreases geometrically:

$$vol(\mathcal{E}^{(k)}) \leq e^{-\frac{k}{2m}} vol(\mathcal{E}^{(0)}).$$

Thus, the algorithm presented either finds a point in the intersection of a finite number of ellipsoids and an ellipsoid which contains that intersection, or determines if the volume of that intersection is smaller than a given value.

3 Deterministic estimation

3.1 Formulation of the set of unfalsified candidates

For the cases where the known set of alternatives is given by an intersection of ellipsoids, we can formalize this set as follows. At each time τ, the set of unfalsified candidates is given by a set of parameters

$$\{\theta \mid \theta^T A(t)\theta - 2\theta^T B(t) + C(t) \leq 0,\ \forall t \in \mathcal{T} \cap [0, \tau]\},$$

where $A(t)$ is positive definite matrix and $\mathcal{T} \subset [0, \tau]$ is a set of time instants [1], [10].

For instance, for the unfalsified control problem with desired behavior given by

$$B_d = \{(r, y, u) \mid \mathcal{J}(r, y, u, \tau) \geq 0\ \forall \tau \in \mathcal{T}\},$$

where r is a reference signal, (y, u) are plant signals, $\mathcal{J}(r, y, u, \tau) \triangleq \Delta(\tau) - \|(y - w_m * r)\|_\tau{}^2$, $w_m = \mathcal{L}^{-1}(W_m(s))$ is a time domain reference model transfer function, Δ is a function $\Delta : \mathcal{T} \to \mathbb{R}_+$, we have that

$$A(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} \overline{w}_{data} \overline{w}_{data}^T dt, \quad (1)$$

$$B(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} \overline{w}_{data} y_{data} dt, \quad \text{and} \quad (2)$$

$$C(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} y_{data}^2 dt - \Delta(\tau), \quad (3)$$

with

$$\overline{w}_{data} = w_m * w_{data} \quad \text{and} \tag{4}$$

$$w_{data} = w(u_{data}, y_{data}) \tag{5}$$

provided that $\|u_{data}\|_\tau + \|y_{data}\|_\tau \neq 0$. The norm utilized $\|x\|_\tau$ is defined by equation 14 and the filtered data $w_{data} = w(u_{data}, y_{data})$ is defined by equation 15.

3.2 Another formulation of the set of unfalsified candidates

Another formulation of the set of unfalsified candidates, is also possible, if we take in consideration the norm of the reference signal when specifying the desired behavior in an unfalsified control problem [10]. Thus, let the desired behavior be given by

$$B_d = \{(r, y, u) \mid \mathcal{J}(r, y, u, \tau) \geq 0 \; \forall \tau \in \mathcal{T}\},$$

where r is a reference signal, (y, u) are plant signals, $\mathcal{J}(r, y, u, \tau) \triangleq \alpha(\tau)\|r\|_\tau{}^2 - \|(y - w_m * r)\|_\tau{}^2$, $w_m = \mathcal{L}^{-1}(W_m(s))$ is a time domain reference model transfer function, α is a function $\alpha : \mathcal{T} \to \mathbb{R}_+$ and $\mathcal{T} \subset [0, \tau]$ is a set of time instants. The set of unfalsified candidates is, in this case, given by the set of parameters

$$\{\theta \mid \theta^T (A(t) - \alpha(t) D(t))\theta - 2\theta^T B(t) + C(t) \leq 0, \; \forall t \in \mathcal{T} \cap [0, \tau]\},$$

where

$$A(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} \overline{w}_{data} \overline{w}_{data}^T dt, \tag{6}$$

$$B(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} \overline{w}_{data} y_{data} dt, \tag{7}$$

$$C(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} y_{data}^2 dt, \quad \text{and} \tag{8}$$

$$D(\tau) = \int_0^\tau e^{-2\sigma(\tau - t)} w_{data} w_{data}^T dt, \tag{9}$$

(10)

with

$$\overline{w}_{data} = w_m * w_{data} \quad \text{and} \tag{11}$$

$$w_{data} = w(u_{data}, y_{data}), \tag{12}$$

provided that $\|u_{data}\|_\tau + \|y_{data}\|_\tau \neq 0$. The norm utilized $\|x\|_\tau$ is defined by equation 14 and the filtered data $w_{data} = w(u_{data}, y_{data})$ is defined by equation 15.

Notice that $A(t) \geq 0$ with the inequality being strict when the disturbance is "persistently exciting" for $\bar{w}(t)$. Consequently, provided we choose $\alpha(t)$ sufficiently small so that $A(t) - \alpha(t)D(t) > 0 \ \forall t \in [0, \tau]$, then the problem is an intersection of ellipsoids exactly as in Section 3.1.

3.3 Complementary Definitions

Definition 3.1 *Given a constant $\sigma > 0$, we define the exponentially-weighted truncated L_2 inner-products $< x, y >_\tau$ and norm $\|x\|_\tau$ by*

$$< x, y >_\tau \triangleq \int_0^\tau e^{-2\sigma(\tau - t)} y^T(t) x(t)\, dt \tag{13}$$

$$\|x\|_\tau \triangleq \sqrt{< x, x >_\tau}. \tag{14}$$

Definition 3.2 *The filter w is defined by*

$$w(u, y) \triangleq (u, v^T(u), y, v^T(y))^T \tag{15}$$

$$\tag{16}$$

where $v : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}^{n-1}$ is given by

$$\dot{v}(q) = \Lambda v(q) + lq \tag{17}$$

$$(v(q))(0) = 0 \tag{18}$$

and (Λ, l) is an asymptotically stable system in controllable canonical form, with

$$\lambda(s) = det(sI - \Lambda) = \lambda_1(s) Z_m(s) \tag{19}$$

for some monic Hurwitz polynomial $\lambda_1(s)$ of degree $n^* - 1$, where n^* is the relative degree of $W_m(s)$ ([10],[17], and [18]).

4 Practical Considerations

The essential idea behind the computational procedure to be used is to apply the ellipsoid algorithm to find an unfalsified candidate which is the center of an ellipsoid that contains the whole set of unfalsified candidates.

Let us notice that at each time instant that we detect a falsification, we can keep computing new ellipsoids of smaller volume until we find one whose center is unfalsified.

Notice that if we have a poor excitation or a specification that is too relaxed we may have an intersection of sets that are not ellipsoids. The essential point to have in mind is that if we have a convex set then there is a cutting plane and if this convex set is an ellipsoid then a cutting plane is very easily computed.

5 Example

In this section, we present simulation results for the unfalsified model reference control problems formulated in section 3. Let us choose $W_m(s) = \frac{1}{s+1} = \frac{s+1}{(s+1)^2}$. Choosing the second form we have that $n = 2$. Additionally, let us choose $\Lambda = -1$, $l = 1$ and $\sigma = 0.01$. The filter w defined in [10] is then given by $w(u,y) = (u, w_m * u, y, w_m * y)^T$ and the class of candidate controllers is given by $\{(\mathbf{U}, \mathcal{B}_c(\theta)) \mid \theta \in \Theta\}$, where $\mathcal{B}_c(\theta) = \{(r,y,u) \mid r = \theta^T w(u,y)\}$ and θ is a constant parameter vector in $\mathbb{R}^4$. For purposes of this simulation let "the true but unknown plant" be given by

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x \end{aligned}$$

where n_1 and n_2 are uncorrelated normally distributed random signals with mean zero and variance one. Let the reference signal $r(t) = signal(cos(0.1\pi t))$ $\forall t \geq 0$. We obtain (y_{data}, u_{data}) by closing the loop with the initial controller associated to the parameter $\theta(0) = \begin{pmatrix} 1 & 0 & 0.1 & -0.1 \end{pmatrix}^T$ and the initial plant state given by $x = 0$. Thus we are able to use our theory to compute a new controller parameter $\theta(t)$ based on the data available at any given time $t = \tau$. Controller adaptation is achieved by repeating this operation periodically as time τ evolves and (y_{data}, u_{data}) accumulates, in order to update the controller parameter θ. This procedure was used to update the controller parameter vector $\theta(t)$ every 5 sec starting at time $\tau = 5$.

For the first problem formulation with $\Delta(t) = 0.1t$, we obtained the simulation results shown in figure 1, while for the second problem formulation with $\alpha(t) = .0001$ we obtained the simulation results shown in figure 2. Let us notice that, we plotted the graphic of the determinant of A_k versus time since the volume of the corresponding ellipsoid is equal to the product of a constant, which depends on the dimension of the parameter space, times the square root of that determinant.

6 Concluding Remarks

An algorithm for computing the intersection of a finite number of ellipsoids was given. This algorithm was used to determine an unfalsified candidates which is the center of an ellipsoid which contains the whole set of unfalsified candidates. Let us notice that if an unfalsified candidate existed, at each time that we detected a falsification, we could keep computing new ellipsoids of smaller volume until we found one whose center was unfalsified.

Let us point out that the ellipsoid algorithm presented here can also contribute to the comparison of classical stochastic estimation and deterministic robust estimation presented in [1]. Let us notice that the cutting plane algorithm presented here besides guaranteeing that we can keep getting closer to the intersection of the ellipsoids as time evolves is guaranteed to find a point in the intersection or determine if the volume of the intersection is smaller than a given value. Thus, it seems to us that the use of convex set and cutting planes should be a salient feature of deterministic robust estimation as compared to classical stochastic estimation.

The application of the algorithm to the solution of the unfalsified model reference adaptive control problem produced a cutting plane algorithm for adaptive control. Let us also notice that the sequence of decreasing volume ellipsoids which contains the set of unfalsified controllers gives a precise characterization of learning in control in terms of a sequence of recursively computable shrinking balls which has been for long looked for by many researchers.

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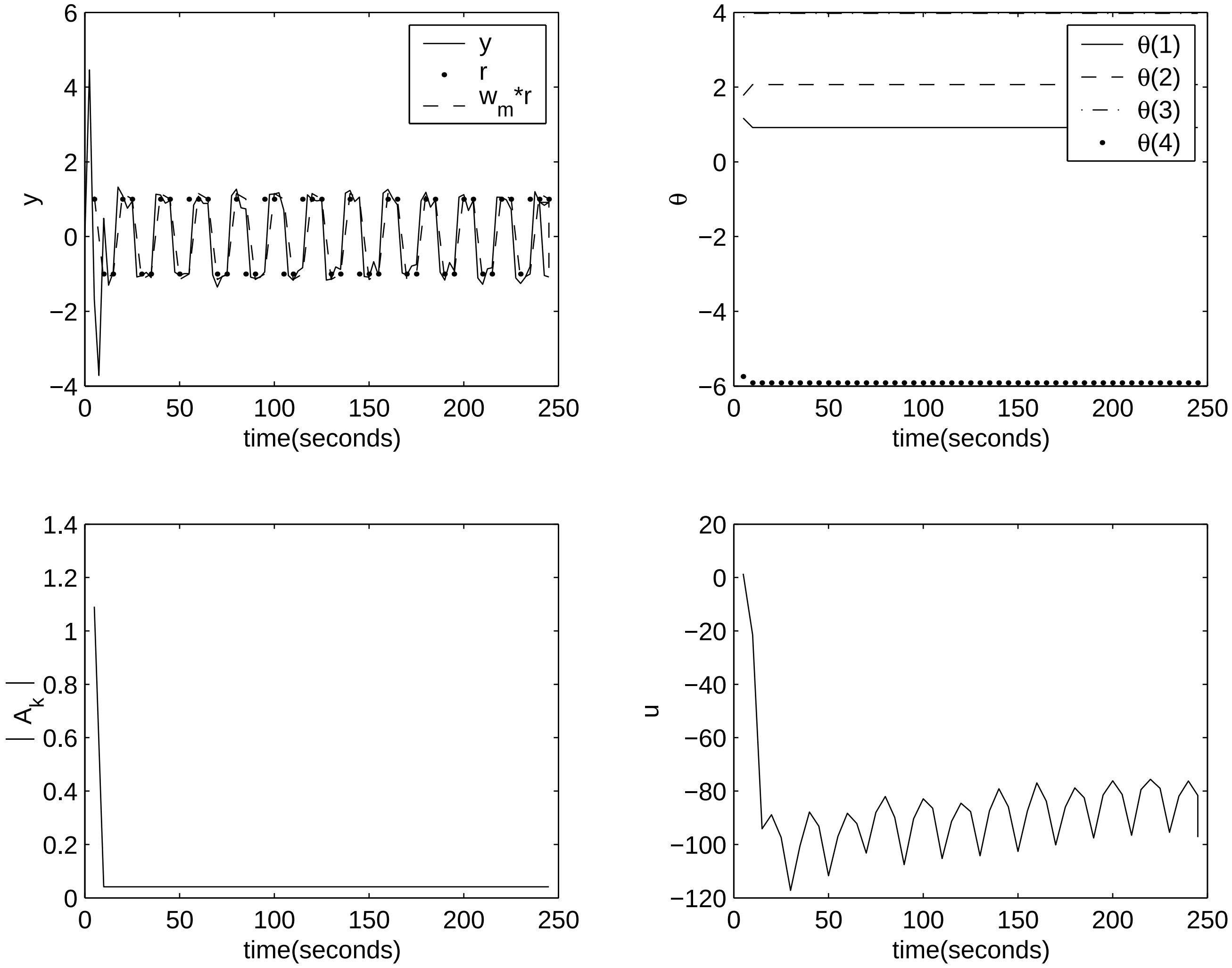

Figure 1: Simulation results for $B_d = \{(r, y, u) \mid \|y - w_m * r\|_\tau{}^2 \leq 0.1\tau \;\; \forall \tau \in \mathcal{T}\}$

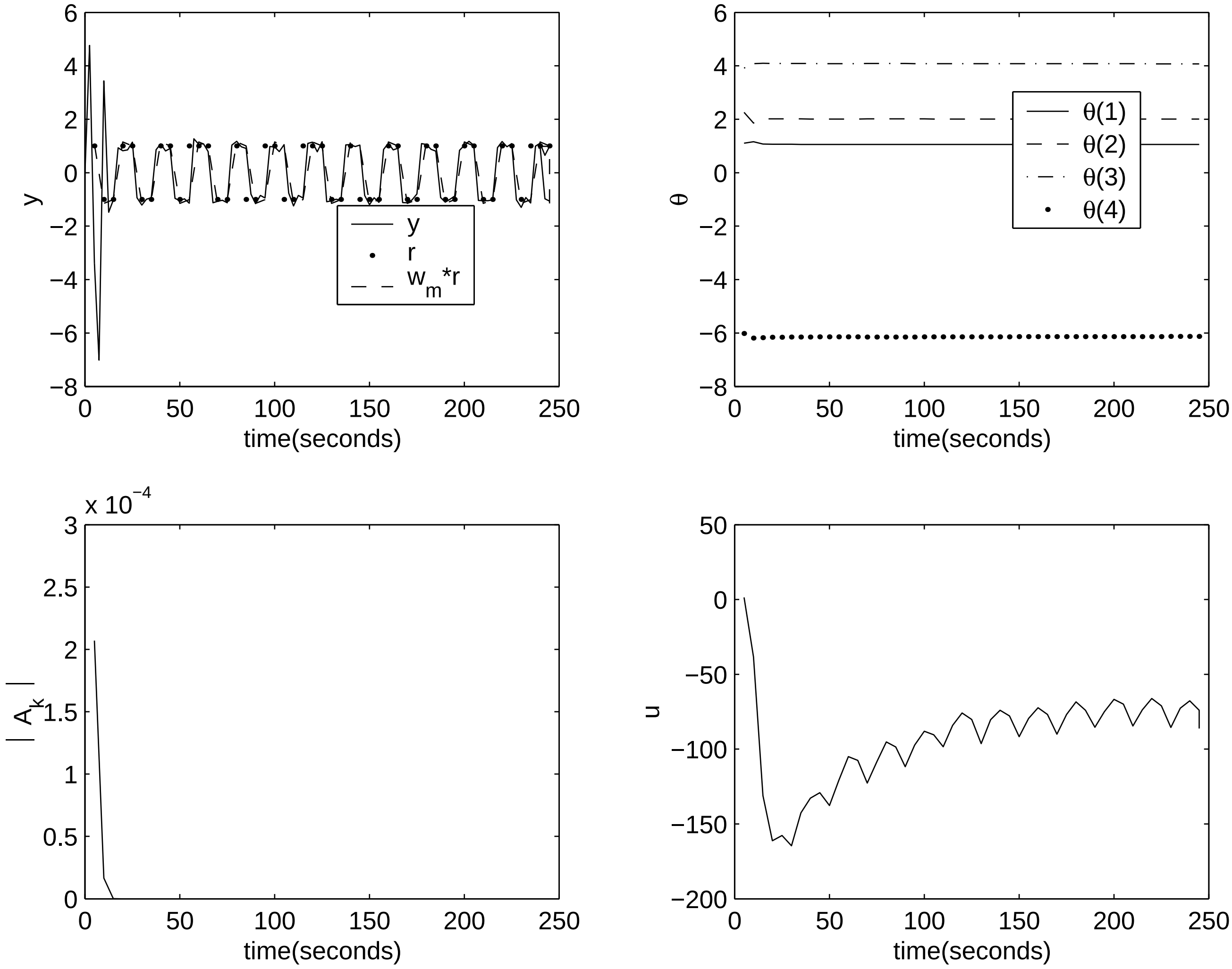

Figure 2: Simulation results for $B_d = \{(r, y, u) \mid \|y - w_m * r\|_\tau{}^2 \leq 0.0001\|r\|_\tau{}^2 \ \forall \tau \in \mathcal{T}\}$