Stability Multipliers for MIMO Monotone Nonlinearities

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Abstract—**For block structured monotone or incrementally positive n-dimensional nonlinearities, the largest class of convolution operators (multipliers) that preserve positivity is derived. These multipliers can be used in conjunction with positivity and IQC stability criteria to evaluate stability and robustness of MIMO feedback systems.**

I. INTRODUCTION

The role of linear time-invariant (LTI) stability multipliers in nonlinear stability theory is to permit less conservative stability conclusions. Such multipliers are useful in conjunction with positivity and related conic-sector and IQC-type stability criteria (cf. [7], [10]). By optimizing over a class of positivity perserving multipliers for a given nonlinearity, one reduces the conservativeness of stability conclusions derived from positivity stability criteria. The key property of such multipliers $\mathcal{M}$ is that they preserve positivity of an incrementally positive nonlinearity N, in the sense that $\mathsf{M} \in \mathcal{M}$ implies positivity of the operator $\mathsf{M}^*\mathsf{N}$. Zames and Falb [4] studied continuous-time SISO feedback systems of the form in Figure 1 for the special case of a single monotone SISO nonlinearity. Going beyond earlier multiplier results of Popov and others that had been derived via the Kalman-Yakubov-Popov (KYP) lemma, they employed input-output approach to derive a remarkably broad class of LTI stability multipliers that preserve positivity of SISO monotone nonlinearities which subsequently was proved to be the entire class of positivity preserving LTI stability multipliers for the SISO case [6], [13].

Table I summarizes the main contributions made by previous researchers, as well as the contributions of this note.

Algorithms for a practical usage of multipliers in the incrementally positive SISO nonlinearities case and the incrementally positive block diagonal repeated SISO case exist (see reference [15] and references therein).

This note focuses on the characterization of the multipliers that preserve positivity of a MIMO monotone or incrementally positive nonlinearity N that may have a particular block structure. These multipliers can be used to prove stability of MIMO discrete time feedback systems as illustrated in Figure 1 and where the following assumptions hold:

A1. $x_i, y_i, e_i,\ i = 1, 2$ are n-dimensional discrete time real signals.

A2. $\mathsf{G} = [\mathsf{G}_{ij}]_{n\times n}$, is a discrete time LTI causal and stable operator.

A3. The block structured nonlinearity in the feedback loop is monotone or incrementally positive.

The results are organized as follows. In Section II, necessary terminology is introduced and background results are stated in Section III. The problems of concern are formally posed in Section IV. Preliminary lemmas are in Section V, main results are presented in Section VI. Finally in section VII main results are briefly discussed and conclusions summarized.

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TABLE I

Main contributions on the characterization of positivity preserving multipliers. (I.P. $\equiv$ Incrementally positive; M. $\equiv$ Monotone.)

Nonlinearity	Continuous time	Discrete time
SISO (M. or I.P.)	[4]	[6]
MIMO (I.P.)	[11]	Thm. 1
MIMO (M.)	[11]	Thm. 1
repeated SISO (I.P. or M.)	[13]	[13]
repeated MIMO (I.P.)	Thm. 3	Thm. 2
repeated (M.)	Thm. 3	Thm. 2

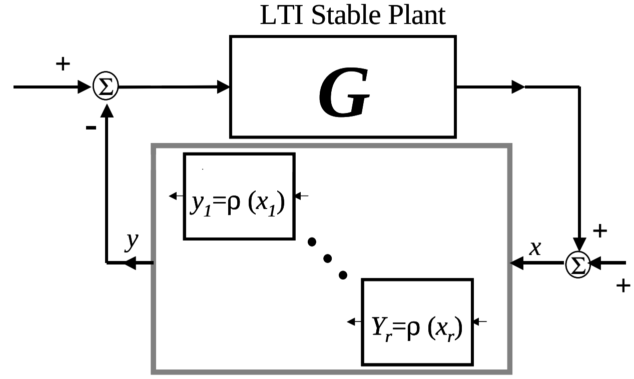

Fig. 1. Repeated MIMO nonlinearities $\rho : \Re^n \to \Re^n$ in feedback loop.

TABLE II
NOTATION.

Symbol	Meaning
$\Re$	Set of real numbers
l_2	Space of infinite sequences $\mathbf{z} = \{z(k)\}$ such that $\sum \|z(k)\|^2$ is summable
l_2^n	Space of infinite sequences $\mathbf{z} = \{z(k)\}$ with $z(k) \in \Re^n$ and $\sum \|z(k)\|^2$ summable
$\mathcal{L}(\mathsf{l}_2^n, \mathsf{l}_2^n)$	Space of linear operators from l_2^n into l_2^n
I_n	$n \times n$ Identity matrix
herm(D)	$\frac{1}{2}(D + \overline{D}')$ for $D \in \Re^{n \times n}$
diag$\{x_1, \ldots, x_n\}$	Diagonal matrix with $x_1, \ldots, x_n$ along its diagonal
$U > 0$	Positive semidefinite matrix U
$\overline{A}$	The complex conjugate
A'	Transpose of A
A^*	$\overline{A}'$ for a matrix A
$\otimes$	Kronecker product of matrices
$Do(\rho)$	Domain of the mapping ρ
$\langle x, y \rangle$	Inner product between sequences x, y
MIMO	Multi-Input-Multi-Output
SISO	Single-Input-Single-Output
LTI	Linear time invariant

II. PRELIMINARIES

The notation used is summarized in Table II.

A few definitions are in order before we can formally pose our problem.

Definition 1 [Incrementally positive mapping, [6]]
Let ρ be a multivalued mapping from $Do(\rho) \subset \Re^n$ into $\Re^n$. Then ρ is said to be *incrementally positive* on $Do(\rho)$ if for all $x(i)$, $x(j) \in Do(\rho)$ one has

$$\langle x(i) - x(j), \widehat{x}(i) - \widehat{x}(j) \rangle \geq 0$$

with $(x(i), \widehat{x}(i))$ and $(x(j), \widehat{x}(j))$ in the graph of ρ.

Definition 2 [Monotone mapping]
A multivalued mapping from $Do(\rho) \subset \Re^n$ into $\Re^n$ will be called *monotone* on $Do(\rho)$ if and only if one has $\rho(x) \subset \partial f(x)$ for every $x \in Do(\rho)$, where f is some closed proper convex function on $\Re^n$.

This definition of monotone mapping is motivated by the results in Safonov et al. [11], which suggest that this is the appropriate extension to n-dimensions of the concept of incrementally positive mapping introduced by Zames and Falb [4]. In the next section, lemma 1 presents a useful characterization of the monotone mappings.

Definition 3 [Doubly hyperdominant matrix, [6]]
A real $(p \times p)$ matrix $M = (m_{kl})$ is said to be *doubly hyperdominant* if

$$m_{kl} \leq 0 \text{ for } k \neq l \;\text{ and }\; \sum_{k=1}^{p} m_{kl} \geq 0,\; \sum_{l=1}^{p} m_{kl} \geq 0 \;\; \forall\, k, l$$

If, additionally,

$$\sum_{k=1}^{p} m_{kl} = \sum_{l=1}^{p} m_{kl} = 0 \;\; \forall\, k, l$$

then M is said to be *doubly hyperdominant with zero excess*.

The definition of an operator $M \in \mathcal{L}(\mathsf{l}_2, \mathsf{l}_2)$ being doubly hyperdominant, is given in terms of its associated array $\{m_{kl}\}$ $k, l \in \mathsf{I}$ and is completely analogous to the one for the matrix M. The only difference is that the subindices k and l run from $-\infty$ to $+\infty$. In the continuous time case, these notions are defined in an analogous manner in terms of the kernel $m(t, \tau)$ of an operator $M \in \mathcal{L}(\mathsf{L}_2, \mathsf{L}_2)$ with integrals replacing sums.

III. BACKGROUND RESULTS

Lemma 1
A multivalued mapping $\rho : Do(\rho) \subset \Re^n$ into $\Re^n$ is monotone if and only if for any permutation π of the integers, and for any sequences $\{x(k)\}, \{\widehat{x}(k)\}$ such that $x(k) \in Do(\rho)$, $\widehat{x}(k) \in \rho(x(k))$, we have

$$\sum_k \langle x(k), \widehat{x}(k) \rangle \geq \sum_k \langle x(k), \widehat{x}(\pi(k)) \rangle \tag{1}$$

Proof:
It is enough to show that the result holds for any cyclic permutation

$$\pi_c : (1, \ldots, q) \rightarrow (q, 1, 2, \ldots, q-2, q-1)$$

since every permutation can be written uniquely (up to a product of factors) as a product of disjoint cycles. This is precisely the result of [5] (theorem 24.8) after a rearrangement of inequality (1). 2

Clearly ρ monotone implies ρ incrementally positive (take π that interchange only two elements). It can be shown, using a rearrangement inequality due to Hardy, Littlewood and Polya [1] (p. 277) that for $n = 1$, the incrementally positive mappings and the monotone mappings are the same. However, when $n > 1$ there exists incrementally positive mappings, which are not monotone. For example, when ρ is the (single valued) linear transformation from $\Re^n$ to $\Re^n$, ρ is monotone if and only if its matrix representation Q is symmetric and positive semidefinite (as may be deduced from the definition of monotone mapping). But ρ is incrementally positive if merely $\langle x(i) - x(j), Q(x(i) - x(j)) \rangle \geq 0$ for all $x(i)$, $x(j)$, in other words if herm$\{Q\}$ is positive semidefinite.

IV. PROBLEM FORMULATION

Problem 1 (Discrete-Time)

Let $\overline{M}$ be an element of $\mathcal{L}(l_2^n, l_2^n)$ which is of the Toeplitz type. Find necessary and sufficient conditions on operator M for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative for any sequences $x = \{x(k)\}$ and $y = \{y(k)\}$, $x(k) \in \Re^n$, $y(k) \in \rho\{x(k)\} \in \Re^n$, $0 \in \rho(0)$, and ρ is a mapping of the following types:

a. Single MIMO block, monotone.

b. Single MIMO block, incrementally positive.

c. Repeated MIMO block, monotone.
($\rho = \text{diag}(\rho_1, \ldots, \rho_r)$ where $\rho_i = \phi$ for all $i = 1, \ldots, r$ and $\phi : \Re^s \to \Re^s$ is monotone, $n = rs$).

d. Repeated MIMO block, incrementally positive.
($\rho = \text{diag}(\rho_1, \ldots, \rho_r)$ where $\rho_i = \phi$ for all $i = 1, \ldots, r$ and $\phi : \Re^s \to \Re^s$ is incrementally positive, $n = rs$).

Problem 2 (Continuous Time)

Let $\overline{M}$ be an element of $\mathcal{L}(L_2^n, L_2^n)$ of the Toeplitz type. Find necessary and sufficient conditions on Toeplitz operator M for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative for any $x, y \in L_2^n$, $y(t) = \rho(x(t))$, $0 \in \rho(0)$ and ρ is a mapping of the following types:

a. Repeated MIMO block, monotone.
($\rho = \text{diag}(\rho_1, \ldots, \rho_r)$ where $\rho_i = \phi$ for all $i = 1, \ldots, r$ and $\phi : \Re^s \to \Re^s$ is monotone, $n = rs$).

b. Repeated MIMO block, incrementally positive.
($\rho = \text{diag}(\rho_1, \ldots, \rho_r)$ where $\rho_i = \phi$ for all $i = 1, \ldots, r$ and $\phi : \Re^s \to \Re^s$ is incrementally positive, $n = rs$).

V. PRELIMINARY RESULTS

Lemma 2 [Generalization of [6], thm. 3.7]

A necessary and sufficient condition for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative for any finite sequences $x = \{x(1), \ldots, x(p)\}$, $y = \{y(1), \ldots, y(p)\}$, $x(i) \in \Re^n$, $y(i) \in \rho(x(i))$, ρ being a monotone (incrementally positive, $n > 1$) mapping, is that the matrix $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} \overline{M}_{11} & \cdots & \overline{M}_{1p} \\ \vdots & \ddots & \vdots \\ \overline{M}_{p1} & \cdots & \overline{M}_{pp} \end{bmatrix} = \begin{bmatrix} m_{11}I_n & \cdots & m_{1p}I_n \\ \vdots & \ddots & \vdots \\ m_{p1}I_n & \cdots & m_{pp}I_n \end{bmatrix}$$

where $M = (m_{kl})$ is a $(p \times p)$ doubly hyperdominant (doubly hyperdominant and symmetric) matrix with zero excess.

Proof of sufficiency:

Let $\overline{M}$ be of the claimed form. Let r be any positive number such that $r \geq m_{kl}$ for all k, l. Clearly $M = r\left[I - \frac{1}{r}(rI - M)\right]$. Now, since $\frac{1}{r}(rI - M)$ is a doubly stochastic matrix, it can be decomposed as $\sum_{i=1}^{N} \alpha_i P_i$ with $\alpha_i \geq 0$, $\sum_{i=1}^{N} \alpha_i = 1$, where P_i is a $(p \times p)$ permutation matrix. Thus M can be written as $M = \sum_{i=1}^{N} \beta_i (I - P_i)$ with $\beta_i \geq 0$, $\beta_i = r\alpha_i$. Therefore, $\overline{M} = M \otimes I_n = \sum_{i=1}^{N} \beta_i (I - P_i) \otimes I_n = \sum_{i=1}^{N} \beta_i (I - \overline{P}_i)$, where $\overline{P}_i = P_i \otimes I_n$. This decomposition of $\overline{M}$ shows that it is enough to prove the claim for the matrices $\left(I - \overline{P}_i\right)$. For the case where ρ is monotone, this is precisely what inequality (1) claims. When ρ is incrementally positive, the result follows directly from the symmetry of $\overline{P}_i$.

Proof of necessity:

Let us assume that $\overline{M}$ is of the form $\overline{M} = M \otimes I_n$, but the matrix M fails to be doubly hyperdominant with zero excess because $m_{kl} > 0$ for some $k \neq l$, in which case the mapping $\rho : x \to y$ is monotone (incrementally positive), as can be seen from Lemma 4, where $x = \{0, \ldots, 0, 1, 0, \ldots, 0\}$, $y = \{0, \ldots, 0, -1, 0, \ldots, 0\}$, and to simplify notation 0, 1 and -1 will denote n-dimensional vectors of zeros, ones and negative ones. Also the 1 and -1 are in the k-th and l-th positions of x and y respectively. Furthermore

$$\langle x, \overline{M}y \rangle = \langle 1, m_{kl} I_n (-1) \rangle = m_{kl}(-n) < 0$$

Assume next, that $\overline{M}$ is of the form $\overline{M} = M \otimes I_n$, but the matrix M fails to be doubly hyperdominant with zero excess, because $\sum_{k=1}^{p} m_{kl} \neq 0$ for some l. The mapping $\rho : x \to y$, where $x = \{1, \ldots, 1, (1 + \varepsilon), 1, \ldots, 1\}$ and $y = \left\{0, \ldots, 0, \frac{1}{\varepsilon}1, 0, \ldots, 0\right\}$ with $\varepsilon \neq 0$, and the elements $(1 + \varepsilon)$ and $\frac{1}{\varepsilon}1$ are in the l-th place of x and y respectively, is monotone (incrementally positive) as can be seen from Lemma 4. This leads to $\langle x, \overline{M}y \rangle = n(\frac{1}{\varepsilon} \sum_{k=1}^{p} m_{kl} + m_{ll})$. Thus, by taking ε sufficiently small and of appropriate sign $\langle x, \overline{M}y \rangle$ can then indeed be made negative.

Assume now, that $\overline{M}$ is not of the form $\overline{M} = M \otimes I_n$.

If $\overline{M}_{kl}$ is not diagonal for some k, l, assume then that the (i, j) (for $i \neq j$) entry of the submatrix $\overline{M}_{kl}$ (denoted by $(\overline{M}_{kl})_{ij}$) is not zero. The mapping $\rho : x \to y$ is monotone (incrementally positive) as follows from lemma 4, where $x = \{0, \ldots, 0, A, 0, \ldots, 0\}$ and $y = \{0, \ldots, 0, B, 0, \ldots, 0\}$ with A and B in the k-th and l-th positions, where $A = (0, \ldots, 0, 1, 0, \ldots, 0)$ and $B = (0, \ldots, 0, \delta, 0, \ldots, 0)$ with 1 and δ in the i-th and j-th positions, respectively. Also,

$$\langle x, \overline{M}y \rangle = \langle A, \overline{M}_{kl}(B) \rangle = \delta(\overline{M}_{kl})_{ij}$$

which can be made negative by choosing δ of the appropriate sign.

If $\overline{M}_{kl}$ is diagonal, but not of the form $m_{kl}I_n$, let us say $(\overline{M}_{kl})_{ii} > (\overline{M}_{kl})_{jj}$ for some $i \neq j$ then, as before, we can check that the mapping $\rho : x \to y$ is monotone (incrementally positive), where

$$x = \{0,\ldots,0,C,0,\ldots,0\}\,,\; y = \{0,\ldots,0,D,0,\ldots,0\}$$

with C, D in the k-th and l-th positions, where

$$C = (0,\ldots,0,-1,0,\ldots,0,1,0,\ldots,0),\; D = (1,\ldots,1)$$

with C having -1 and 1 in the i-th, j-th positions, thus

$$\langle x, \overline{M}y\rangle = \langle C, \overline{M}_{kl}(D)\rangle = (\overline{M}_{kl})_{jj} - (\overline{M}_{kl})_{ii} < 0$$

A similar argument works in the case $(\overline{M}_{kl})_{ii} < (\overline{M}_{kl})_{jj}$.

Finally, assume that $\overline{M} = M \otimes I_n$ and M is doubly hyperdominant, but not symmetric. $\overline{M} = \sum_{i=1}^{N} \beta_i \left(I - \overline{P}_i\right)$, $\beta_i \geq 0$, where $\overline{P}_i = P_i \otimes I_n$, P_i permutation matrices $\overline{M} = \sum_{i=1}^{N} \beta_i (I - \overline{P}_i)$, $\beta_i \geq 0$

$$\overline{M} = \sum_{i=1}^{N} \beta_i (I - \overline{P}_i),\; \beta_i \geq 0$$

where $\overline{P}_i = P_i \otimes I_n$, and P_i are permutation matrices

can be decomposed as

$$\overline{M} \equiv \overline{M}_1 + \overline{M}_2 \equiv \sum_{i=1}^{N_1} \beta_i^s (I - \overline{P}_i^s) + \sum_{j=1}^{N_2} \beta_j^{ns} (I - \overline{P}_j^{ns})$$

with $N_1 + N_2 = N$, $N_2 \geq 1$

where $\overline{M}_1, \overline{M}_2$ are doubly hyperdominat with zero excess and $\overline{M}_1$ is symmetric, $\overline{M}_2$ is not symmetric and $(\overline{M}_2)_{ij} = 0$ or $(\overline{M}_2)_{ji} = 0$ for $j \neq i$. This implies in particular that the permutations P_i^s and P_j^{ns} can be chosen to be symmetric and not symmetric matrices respectively. Choose any P_j^{ns}, say $P_{\hat{j}}^{ns}$ then, there is at least a cyclic permutation P_c of size $\nu \geq 3$, such that for easy of notation we write $P_{\hat{j}}^{ns} = P_c \cup P$ for some permutation P. Where P_c is the following $\nu \times \nu$ matrix

$$P_c = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & \ddots & & \\ & & 1 & \end{bmatrix}$$

We can verify that the mapping ρ that sends $x \mapsto y$ is incrementally positive (but not monotone), where $x = \{A,0,\ldots,0,B,0,\ldots,0\}$ and $y = \{-B,,0,\ldots,0,A,0,\ldots,0\}$, $A = (1,0,\ldots,0)$, $B = (0,1,0,\ldots,0,0)$ with B, A in the ν position of x and y respectively. Furthermore

$$\langle x, \overline{M}y\rangle =$$

$$= \left\langle x, (\sum_{i=1}^{N_1} \beta_i^s (I - \overline{P}_i^s) + \sum_{j=1}^{N_2} \beta_j^{ns} (I - \overline{P}_j^{ns}))y \right\rangle =$$

$$= \left\langle x, \sum_{j=1}^{N_2} \beta_j^{ns} (I - \overline{P}_j^{ns})y \right\rangle$$

Also notice that there is no j such that the permutation $\overline{P}_j^{ns}$ has a one in the $(\nu, 1)$ position, because $(\overline{M}_2)_{ij} = 0$ or $(\overline{M}_2)_{ji} = 0$ for $j \neq i$. Thus

$$\left\langle x, \sum_{j=1}^{N_2} \beta_j^{ns} (I - \overline{P}_j^{ns})y \right\rangle \leq$$

$$\leq \beta_c^{ns} \langle x, (I - P_c)y\rangle = -\beta_c^{ns} < 0$$

Therefore M must be symmetric. 2

Under the additional assumption that $0 \in \rho(0)$, we can eliminate the zero excess condition in lemma 2:

Lemma 3 [Generalization of [6], thm. 3.8]

A necessary and sufficient condition for the inner product $\langle x, \overline{M}y\rangle$ to be nonnegative for any sequences $x = \{x(1),\ldots,x(p)\}$, $y = \{y(1),\ldots,y(p)\}$, $x(i) \in \Re^n$, $y(i) \in \rho(x(i)) \in \Re^n$, ρ being a monotone (incrementally positive, $n > 1$) mapping, such that $0 \in \rho(0)$ is that the matrix $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} M_{11} & \cdots & M_{1p} \\ \vdots & \ddots & \vdots \\ M_{p1} & \cdots & M_{pp} \end{bmatrix} = \begin{bmatrix} m_{11}I_n & \cdots & m_{1p}I_n \\ \vdots & \ddots & \vdots \\ m_{p1}I_n & \cdots & m_{pp}I_n \end{bmatrix}$$

where $M = (m_{kl})$ is a $(p \times p)$ doubly hyperdominant (doubly hyperdominant and symmetric) matrix.

Proof :

After augmenting the vectors x and y with zero the result follows from lemma 2 after some work. 2

Due to the result of lemma 3, in the sequel we assume that $\rho : \Re^n \to \Re^n$ is a multivalued mapping with $0 \in \rho(0)$.

The following extension to Lemma 3 hold.

Lemma 4 [Generalization of [6], thm. 3.11]

Let $\overline{M} \in \mathcal{L}(l_2^n, l_2^n)$. A necessary and sufficient condition for the inner product $\langle x, \overline{M}y\rangle$ to be nonnegative for any sequences $x = \{x(k)\}$, $y = \{y(k)\}$, $x(i) \in \Re^n$, $y(i) \in \rho(x(i)) \in \Re^n$, ρ being a monotone (incrementally positive, $n > 1$) mapping is that $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & M_{11} & \cdots & M_{1p} & \cdots \\ & \vdots & & \vdots & \\ \cdots & M_{p1} & \cdots & M_{pp} & \cdots \\ & \vdots & & \vdots & \end{bmatrix} =$$

$$= \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & m_{11}I_n & \cdots & m_{1p}I_n & \cdots \\ & \vdots & & \vdots & \\ \cdots & m_{p1}I_n & \cdots & m_{pp}I_n & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

where $M = (m_{kl})$ is doubly hyperdominant (doubly hyperdominant and symmetric).

Proof:

By the previous Lemma 3, all finite truncations of the infinite sum in the inner product $\langle x, \overline{M}y \rangle$ yield a nonnegative number. Thus the limit, since it exists, is also nonnegative. 2

VI. MAIN RESULTS

In the particular case that $\overline{M}$ represents a convolution operator. We have:

Answer to Problem 1, parts a and b :

Theorem 1 [Generalization of [6], thm. 3.12]

Let $\overline{M}$ be an element of $\mathcal{L}(\mathrm{l}_2^n, \mathrm{l}_2^n)$, which is of the Toeplitz type. Then a necessary and sufficient condition for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative for any sequences $x = \{x(k)\}$, $y = \{y(k)\}$, $x(k) \in \Re^n$, $y(k) \in \rho\{x(k)\} \in \Re^n$, ρ being a monotone (incrementally positive, $n > 1$) mapping, is that $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & M_0 & \cdots & M_p & \cdots \\ & \vdots & & \vdots & \\ \cdots & M_{-p} & \cdots & M_0 & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & m_0 I_n & \cdots & m_p I_n & \cdots \\ & \vdots & & \vdots & \\ \cdots & m_{-p} I_n & \cdots & m_0 I_n & \cdots \\ & \vdots & & \vdots & \end{bmatrix} \quad (3)$$

where $M = (m_{kl}) = (m_{k-l})$ is doubly hyperdominant and Toeplitz (symmetric, doubly hyperdominant and Toeplitz).

Proof:

Special case of Lemma 4 . 2

Remark

Theorem 1 gives a necessary and sufficient MIMO generalization (discrete time) of the sufficient SISO stability conditions derived by Zames-Falb [4], thm. 2 (continuous time). The sufficiency conditions in [4] were generalized to the MIMO case by Safonov-Kulkarni [11], thm. 1, (continuous time) again as sufficient conditions only.

Answer to Problem 1, parts c and d :

Theorem 2

Let $\overline{M}$ be an element of $\mathcal{L}(\mathrm{l}_2^n, \mathrm{l}_2^n)$, which is of the Toeplitz type. A necessary and sufficient condition for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative for any sequences $x = \{x(k)\}$, $y = \{y(k)\}$, $x(k) \in \Re^n$, $y(k) = \rho\{x(k)\} \in \Re^n$, generated by r-fold repeated nonlinearity $\rho = \mathrm{diag}\,(\rho_1, \ldots, \rho_r)$ with $\rho_i = \phi$ for all $i = 1, \ldots, r$ and $\phi : \Re^s \to \Re^s$ being a monotone (incrementally positive, $n > 1$) mapping is that $\overline{M}$ be of the form

$$\overline{M} = \{M_{kl}\}$$

where

$$M_{kl} = \begin{bmatrix} m_{11}^{k-l} I_s & \cdots & m_{1r}^{k-l} I_s \\ \vdots & \ddots & \vdots \\ m_{r1}^{k-l} I_s & \cdots & m_{rr}^{k-l} I_s \end{bmatrix} \quad (4)$$

and the array $\{M_{kl}\}$ is Toeplitz (Toeplitz and symmetric) and doubly hyperdominant.

Proof:

Let $x(k) = (x_1(k), \ldots, x_r(k))$ and $y(k) = (y_1(k), \ldots, y_r(k))$ where $y_i(k) = \phi(x_i(k))$. Then from Lemma 4, we conclude that a necessary and sufficient condition for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative is that $\overline{M} = \{M_{kl}\}$ with

$$M_{kl} = \begin{bmatrix} \tilde{m}_{11}^{k,l} I_s & \cdots & \tilde{m}_{1r}^{k,l} I_s \\ \vdots & \ddots & \vdots \\ \tilde{m}_{r1}^{k,l} I_s & \cdots & \tilde{m}_{rr}^{k,l} I_s \end{bmatrix} \quad (5)$$

is doubly hyperdominant (symmetric, doubly hyperdominant and Toeplitz). Adding the requirement that the $\{M_{kl}\}$ be of the Toeplitz type requires that $\tilde{m}_{ij}^{k,l} = m_{ij}^{k-l}$ for all $i, j = 1, ..., r$ as in (4). 2

Theorem 3

Let $\overline{M}$ be an element of $\mathcal{L}(\mathrm{L}_2^n, \mathrm{L}_2^n)$ which is of the Toeplitz type having impulse response

$$\widehat{M}(t) = M_0 \delta(t) - H(t) \quad (6)$$

with $H(t) \in\in \mathrm{L}_2^{n\times n}$ is continous, and $M_0 \in \Re^{n\times n}$.

Then a necessary and sufficient condition for the inner product $\langle x, \overline{M}y \rangle$ to be nonnegative for any signals $x \in \mathrm{L}_2^n$, $y \in \mathrm{L}_2^n$, $y(t) = \rho\{x(t)\} \in \Re^n$, generated by r-fold repeated nonlinearity $\rho = \mathrm{diag}\,(\rho_1, \ldots, \rho_r)$ with $\rho_i = \phi$ for all $i = 1, \ldots, r$ and $\phi : \Re^s \to \Re^s$ being a monotone (incrementally positive, $n > 1$) mapping are that M_0 be of the form

$$M_0 = \mathrm{diag}(m_1 I_s, \ldots, m_r I_s)$$

and

$$H(t) = \begin{bmatrix} h_{11}(t)I_s & \cdots & h_{1r}(t)I_s \\ \vdots & \ddots & \vdots \\ h_{r1}(t)I_s & \cdots & h_{rr}(t)I_s \end{bmatrix}$$

with $h_{ij}(t) \geq 0$ for all $i, j = 1, \ldots, r$ and that the matrix

$$M_0 - \int_{-\infty}^{\infty} H(t)\ dt \ \in \ \Re^{n \times n} \qquad (7)$$

be doubly hyperdominant.

Proof:

We shall discretize the signals $x(t), y(t)$ and the operator $\overline{M} \in \mathcal{L}(\mathsf{L}_2^n, \mathsf{L}_2^n)$ by sampling the signals x, y with sampling interval ϵ. Let $x_{d,\epsilon}(k) = \sqrt{\epsilon}x_{c,\epsilon}(k\epsilon)$, $y_{d,\epsilon}(k) = \sqrt{\epsilon}y_{c,\epsilon}(k\epsilon)$. Then

$$\begin{aligned} \langle x, \overline{M}y \rangle &= \int_{-\infty}^{\infty} x'(t) \int_{-\infty}^{\infty} \widehat{M}(t-\tau)y(\tau)\ d\tau\ dt \\ &= \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_{d,\epsilon}(k) M_{kl} y_{d,\epsilon}(l) \end{aligned}$$

where

$$M_{kl} = \begin{cases} \frac{1}{\epsilon} M_0, & \text{if } k = l \\ -\widehat{H}(\epsilon(k-l)), & \text{otherwise.} \end{cases}$$

The condition (7) ensures that, in the limit as $\epsilon \to 0$, $\{M_{kl,\epsilon}\}$ is doubly hyperdominant. The result follows immediately from Theorem 2. 2

Remark The main contributions of this note are presented in Table I. It is perhaps noteworthy that all the results shown in this table are in fact special cases of our main results Theorems 2 and 3.

VII. CONCLUSIONS

The concepts of monotonic and incrementally positive MIMO mappings have been stated clearly, and shown to be different in general. The entire set of LTI multiplier matrices that preserve positivity of a repeated MIMO monotone and incrementally positive nonlinearities has been characterized. Both the discrete time case and continuous time cases have been treated, and previously known multipliers derived by Zames and Falb and others emerge as special cases.

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