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Fitting Controllers to Data: The MIMO Case

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Abstract

The problem of optimally fitting controllers to data is examined for the identification of a controller from a given class of MIMO controllers used in model reference adaptive control. The problem of identifying a MIMO controller from this class is formulated. This formulation leads to an optimization problem where a best $2nm \times m$ matrix of parameters is looked for. It is proved that the solution to this problem can be given in terms of the solution of an optimization problem where a best $2nm \times 1$ vector of parameters is looked for, which has already been solved for the SISO case. Simulations are provided for an example which has appeared a few times in the model reference adaptive control literature. The formulation and solution of this problem illustrates an unifying link between the design of model reference adaptive control for SISO and MIMO linear systems.

Keywords: multivariable systems; fitting; behavioral approach; model reference; adaptive control.

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1 Introduction

The design of model reference adaptive control for MIMO linear systems has been considered in several works for the past few years ([1],[2],[3],[4],[5]). However, differently from what happens with SISO linear systems, where we have "a parameter estimation perspective for model reference adaptive control" [6], the identification step for MIMO linear systems has been dealt with, in many situations, through the application of arbitrary identification algorithms.

In this work, we show that it is possible to deal with MIMO controllers in an way analogous to the way SISO controllers were dealt with in [7]. More, specifically, we examine the problem of fitting controllers to data for one class of MIMO controllers used in model reference adaptive control.

This formulation leads to an optimization problem where a best $2nm \times m$ matrix of parameters is looked for. It is proved that the solution to this problem can be given in terms of the solution of an optimization problem where a best $2nm^2 \times 1$ vector is looked for, which has already been solved for the SISO case [7].

2 Basic Definitions

The problem of optimally fitting controllers to data was defined in [7] according to the Willems' behavioral approach to dynamical systems [8]. Two basic definitions of the behavioral framework were used, namely the definition of a mathematical model and the definition of a data set.

Definition 2.1 *A mathematical model is a pair* $(\mathbf{U}, \mathscr{B})$*, with* $\mathbf{U}$ *the universum — its elements are called outcomes — and* $\mathscr{B} \subseteq \mathbf{U}$ *the behavior.*

Definition 2.2 *A data set is a nonempty subset* $\mathscr{D}$ *of* $\mathbf{U}$.

Using these two definitions, defining a controller as a mathematical model, noticing that the intersection of behaviors is a way of additional restrictions on a system, an optimization problem was formulated with the goal of finding a best controller from a given class.

Definition 2.3 *A controller is a mathematical model.*

Problem 2.1 *Given a class of controllers* $\{(\mathbf{U}, \mathscr{B}_c(\theta)) \mid \theta \in \Theta\}$*, where* Θ *is a set of parameter vectors, the performance (cost) index* $\mathscr{J}_\tau$*, the operator* $\mathscr{E}$*, the time truncation operator* P_τ*,* $\tau \in \mathbb{R}_+$*, and a data set* $\mathscr{D}_\tau \subset P_\tau \mathbf{U}$*, find the set of parameters* Θ^* *such that*

$$\Theta^*(\tau) = \arg\min_{\theta \in \Theta} \mathscr{J}_\tau(\theta) \tag{1}$$

where

$$\mathscr{J}_\tau(\theta) \stackrel{\Delta}{=} \mathscr{E}(\{\mathscr{I}_\tau(b) \mid b \in \mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta))\}).$$

and $\mathscr{E}$ *denotes either the mean, the max, or the expectation operator and* $\mathscr{I}_\tau$ *is a functional*

$$\mathscr{I}_\tau : \mathbf{P}_\tau(\mathbf{U}) \to \mathbb{R}.$$

3 Problem Formulation

3.1 The System

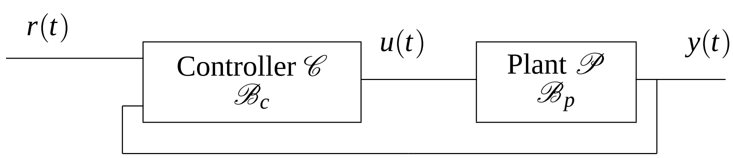

Figure 1: Feedback control system.

Consider the system in Figure 1. As in [7], we omit any arrows on the block diagram in Figure 1. This illustrates a departure from the usual input/output setting, from the processor point of view.

3.2 The Universum

Let $z = (r, y, u) \in \mathbf{U}$ where $\mathbf{U} = \mathscr{R} \times \mathscr{Y} \times \mathscr{U} = \mathscr{L}_{2e}^{n_z}$. Here $\mathscr{R} = \mathscr{L}_{2e}^{n_r}$ is the set of reference signals, $\mathscr{Y} = \mathscr{L}_{2e}^{n_y}$ and $\mathscr{U} = \mathscr{L}_{2e}^{n_u}$ are sets of plant signals, and $n_z = n_r + n_y + n_u$. In this paper we focus on the case $n_r = n_y = n_u = m$.

3.3 The Data Set

The plant information imposes restrictions only on the past values of the signals u and y. Thus the data set is given by

$$\mathscr{D}_\tau = \{(r, y, u) \in \mathbf{P}_\tau(\mathbf{U}) \mid y = y_{data}, u = u_{data}\},$$

for some

$$(y_{data}, u_{data}) \in \mathbf{P}_\tau(\mathscr{Y} \times \mathscr{U}).$$

3.4 The Performance (Cost) Index

Before introducing our performance index, let us define the norm to be used.

Definition 3.1 *Given a constant $\sigma > 0$, we define the exponentially-weighted truncated L_2 inner-products $< x,y >_\tau$ and norm $\|x\|_\tau$ by*

$$< x,y >_\tau \stackrel{\Delta}{=} \int_0^\tau e^{-2\sigma(\tau-t)} y^T(t)x(t)\, dt \tag{2}$$

$$\|x\|_\tau \stackrel{\Delta}{=} \sqrt{< x,x >_\tau}. \tag{3}$$

Given a proper stable transfer function matrix $M_0(s) \in \Re^{m\times m}(s)$ and

$$H(s) = \begin{bmatrix} \frac{1}{(s+1)^{r_1}} & 0 & \cdot & \cdot \\ \frac{1}{(s+1)^{r_2-1}} & \frac{1}{(s+1)^{r_2}} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{(s+1)^{r_m}} \end{bmatrix}.$$

Let the performance (cost) index be given by

$$\mathscr{I}_\tau((r,y,u)) \stackrel{\Delta}{=} \begin{cases} \|y - HM_0[r]\|_\tau^2 / \|M_0[r]\|_\tau^2, & \text{if } \|M_0[r]\|_\tau \neq 0 \\ 0, & \text{if } \|M_0[r]\|_\tau = 0 \text{ and } \|y\|_\tau = 0 \\ \infty, & \text{otherwise.} \end{cases}$$

3.5 The Class of Candidate Controllers

The class of candidate controllers is similar to the one used in multivariable model reference adaptive control ([3],[4],[5]) and also similar to the one used in the [9] and [10] for the SISO case. In order to define the class of candidate controllers, we first define a vector of filters as in ([7],[11]). Notice that we define a vector of filters, and not just a vector of time-domain filtered signals as in [9] and [10]. Let us define $v : \mathscr{L}_{2e}^m \to \mathscr{L}_{2e}^{m(n-1)}$ by

$$\dot{v}(q) = \begin{bmatrix} \Lambda & 0 & \cdot & 0 \\ 0 & \Lambda & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \Lambda \end{bmatrix} v(q) + \begin{bmatrix} l & 0 & \cdot & 0 \\ 0 & l & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & l \end{bmatrix} q \tag{4}$$

$$(v(q))(0) = 0 \tag{5}$$

where (Λ, l) is an asymptotically stable system in controllable canonical form, with

$$\lambda(s) = det(sI - \Lambda) \tag{6}$$

for some monic Hurwitz polynomial $\lambda(s)$ of degree $n-1$. Let us also define

$$w(u,y) \quad = \quad (u^T, v^T(u), y^T, v^T(y))^T. \tag{7}$$

The class of controllers considered is given by $\{(\mathbf{U}, \mathscr{B}_c(\theta)) \mid \theta \in \Theta\}$, where

$$\mathscr{B}_c(\theta) = \{(r,y,u) \mid M_0[r] = \theta^T w(u,y)\} \ \text{ and}$$

Θ is the set of $2nm \times m$ matrix of real-valued constant parameters.

4 Problem Solution

4.1 The Set of Optimal Controllers

Theorem 4.1 *The set of parameter matrices* $\Theta^*(\tau)$,

$$\Theta^*(\tau) = \arg\min_{\theta \in \mathbb{R}^{m\times 2nm}} \mathscr{E}\{\mathscr{I}_\tau(b) \mid b \in P_\tau(P_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta))\}$$

is given by

$$\Theta^*(\tau) = \left\{ \begin{bmatrix} \overline{\theta^*}_1 & \cdot & \cdot & \overline{\theta^*}_m \end{bmatrix} \mid \overline{\theta^*} = \begin{bmatrix} \overline{\theta^*}_1 \\ \cdot \\ \cdot \\ \overline{\theta^*}_m \end{bmatrix} = \arg\min_{\overline{\theta} \in \mathbb{R}^{2nm^2}} \left\{ \frac{\overline{\theta}^T A(\tau)\overline{\theta} - 2\overline{\theta}^T B(\tau) + C(\tau)}{\overline{\theta}^T D(\tau)\overline{\theta}} \right\} \right\}, \tag{8}$$

where

$$A(\tau) = \begin{bmatrix} A_{11}(\tau) & A_{12}(\tau) & \cdot & A_{1m}(\tau) \\ A_{21}(\tau) & A_{22}(\tau) & \cdot & A_{2m}(\tau) \\ \cdot & \cdot & \cdot & \cdot \\ A_{m1}(\tau) & A_{m2}(\tau) & \cdot & A_{mm}(\tau) \end{bmatrix},$$

$$B(\tau) = \begin{bmatrix} B_1(\tau) \\ B_2(\tau) \\ \cdot \\ B_m(\tau) \end{bmatrix},$$

$$D(\tau) = \begin{bmatrix} D_{11}(\tau) & 0 & \cdot & 0 \\ 0 & D_{22}(\tau) & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & D_{mm}(\tau) \end{bmatrix},$$

with

$$
\begin{aligned}
A_{jl}(\tau) &= \int_0^\tau e^{-2\sigma(\tau-t)} \sum_{i=\max\{j,l\}}^{m} H_{ij}[w_{data}](H_{il}[w_{data}])^T dt, && (9)\\
B_j(\tau) &= \int_0^\tau e^{-2\sigma(\tau-t)} \sum_{i=j}^{m} y_{data,i} H_{ij}[w_{data}] dt, && (10)\\
C(\tau) &= \int_0^\tau e^{-2\sigma(\tau-t)} y_{data}^T y_{data} dt, && (11)\\
D_{ij}(\tau) &= \int_0^\tau e^{-2\sigma(\tau-t)} w_{data} {w_{data}}^T dt \textit{ for } i=j, && (12)\\
&= 0 \textit{ otherwise}, \quad \textit{and} && (13)
\end{aligned}
$$

$$
w_{data} = w(u_{data}, y_{data}) \tag{14}
$$

provided that $\|u_{data}\|_\tau + \|y_{data}\|_\tau \neq 0$.

Proof. Let us prove, first, that $\mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta))$ is a unitary set (i.e., it has one and only one point):

$$
\begin{aligned}
\mathscr{D}(\tau) &= \{(r,y,u) \in \mathbf{P}_\tau(\mathbf{U}) \mid y = y_{data}, u = u_{data}\}\\
\mathbf{P}_\tau^{-1}(\mathscr{D}_\tau) &= \{(r,y,u) \in \mathbf{U} \mid \mathbf{P}_\tau(y) = y_{data}, \mathbf{P}_\tau(u) = u_{data}\}\\
\mathbf{P}_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta) &= \{(r,y,u) \in \mathbf{U} \mid r = \theta^T w(u,y), \mathbf{P}_\tau(y) = y_{data}, \mathbf{P}_\tau(u) = u_{data}\}\\
\mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta)) &= \{(r,y,u) \in \mathbf{P}_\tau(\mathbf{U}) \mid r = \theta^T w_{data}, y = y_{data}, u = u_{data}\}.
\end{aligned}
$$

where w_{data} is defined by the equation (14) and by the equations (4) to (5).

Thus $\mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta))$ is a unitary set, which implies that we can restrict ourselves to the problem of finding the set of parameters $\Theta^*(\tau)$ such that

$$
\begin{aligned}
\Theta^*(\tau) &= \arg\min_{\theta\in\Theta}\{\mathscr{I}_\tau(b) \mid b \in P_\tau(P_\tau^{-1}(\mathscr{D}_\tau) \cap \mathscr{B}_c(\theta))\}\\
&= \arg\min_{\theta\in\Theta}\{\frac{\|y_{data} - H[\theta^T w_{data}]\|_\tau{}^2}{\|\theta^T w_{data}\|_\tau{}^2}\}. && (15)
\end{aligned}
$$

Let us define $\theta_1, \ldots, \theta_m$ by

$$
\theta^T = \begin{bmatrix} \theta_1^T \\ \cdot \\ \cdot \\ \theta_m^T \end{bmatrix}.
$$

Then

$$M_0[r] = \theta^T w_{data} = \begin{bmatrix} \theta_1^T w_{data} \\ \cdot \\ \cdot \\ \theta_m^T w_{data} \end{bmatrix}$$

and, consequently,

$$HM_0[r] = \begin{bmatrix} \sum_{j=1}^m H_{1j}[\theta_j^T w_{data}] \\ \cdot \\ \cdot \\ \sum_{j=1}^m H_{mj}[\theta_j^T w_{data}] \end{bmatrix},$$

$$\begin{aligned}
(HM_0[r])^T HM_0[r] &= \sum_{i=1}^m \sum_{j=1}^m \sum_{l=1}^m H_{ij}[\theta_j^T w_{data}](H_{il}[\theta_l w_{data}])^T \\
&= \sum_{i=1}^m \sum_{j=1}^m \sum_{l=1}^m \theta_j^T H_{ij}[w_{data}](H_{il}[w_{data}])^T \theta_l \\
&= \sum_{j=1}^m \sum_{l=1}^m \theta_j^T (\sum_{i=1}^m H_{ij}[w_{data}](H_{il}[w_{data}])^T)\theta_l \\
&= \sum_{j=1}^m \sum_{l=1}^m \theta_j^T (\sum_{i=\max\{j,l\}}^m H_{ij}[w_{data}](H_{il}[w_{data}])^T)\theta_l,
\end{aligned}$$

$$\begin{aligned}
y_{data}^T HM_0[r] &= \sum_{i=1}^m y_{data,i} \sum_{j=1}^m H_{ij}[\theta_j^T w_{data}] \\
&= \sum_{j=1}^m \theta_j^T \sum_{i=1}^m y_{data,i} H_{ij}[w_{data}] \\
&= \sum_{j=1}^m \theta_j^T \sum_{i=j}^m y_{data,i} H_{ij}[w_{data}].
\end{aligned}$$

Let us define

$$\overline{\theta} \triangleq \begin{bmatrix} \theta_1 \\ \cdot \\ \cdot \\ \theta_m \end{bmatrix}, \text{ and } A(\tau),\ B(\tau),\ C(\tau),\ D(\tau),$$

as in the statement of the theorem. We can, then, state that the following equality holds

$$\frac{\|y_{data} - H[\theta^T w_{data}]\|_\tau{}^2}{\|\theta^T w_{data}\|_\tau{}^2} = \frac{\overline{\theta}^T A(\tau)\overline{\theta} - 2\overline{\theta}^T B(\tau) + C(\tau)}{\overline{\theta}^T D(\tau)\overline{\theta}}.$$

From this equality and the expression 15, the thesis follows. □

4.2 Matrix Properties

Property 4.1 *The null space of $D(\tau)$ is contained in the null space of $A(\tau)$ and in the null space of $B^T(\tau)$.*

Proof. We have that

$$\begin{aligned}
D(\tau)\overline{\theta} = 0 & \\
\Rightarrow &\quad D_{jj}(\tau)\theta_j = 0 \\
\Rightarrow &\quad \theta_j^T D_{jj}(\tau)\theta_j = 0 \\
\Rightarrow &\quad \|\theta_j^T w_{data}\|_\tau{}^2 = 0 \\
\Rightarrow &\quad \|H_{ij}[\theta_j^T w_{data}]\|_\tau{}^2 = 0
\end{aligned}$$

which implies that

$$\begin{aligned}
&A(\tau)\overline{\theta} = 0 \;\text{ and} \\
&B^T(\tau)\overline{\theta} = 0.
\end{aligned}$$

□

Property 4.2 *The matrices $D(\tau)$ and*

$$\begin{bmatrix} A(\tau) & -B(\tau) \\ -B^T(\tau) & C(\tau) \end{bmatrix}$$

are symmetric and positive semidefinite.

Proof. A simple inspection reveals that these matrices are symmetric matrices. The positive semidefiniteness of these matrices follows by observing that

$$\begin{bmatrix} \theta \\ 1 \end{bmatrix}^T \begin{bmatrix} A(\tau) & -B(\tau) \\ -B^T(\tau) & C(\tau) \end{bmatrix} \begin{bmatrix} \theta \\ 1 \end{bmatrix} = \|y_{data} - H[\theta^T w_{data}]\|_\tau{}^2 \geq 0$$

$$\theta^T D(\tau)\theta = \sum_{j=1}^{m} \|\theta_j^T w_{data}\|_\tau{}^2 \geq 0$$

$\square$

These properties together with the theorem proved imply that the optimization problem derived from the problem of fitting MIMO controllers to data can be reduced to an optimization problem derived from a problem of fitting SISO controllers to data. This problem is solved in [7].

5 Example

Let us choose

$$M_0(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and}$$

$$H(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix},$$

as in ([3],[4],[5]). Let us also choose $\Lambda = -1$, $l = 1$ and $\sigma = 0.01$. The filter w defined in section 3.5 is then given by $w(u,y) = (u^T, \frac{1}{s+1}[u^T], y^T, \frac{1}{s+1}[y^T])^T$ and the class of candidate controllers is given by $\{(\mathbf{U}, \mathscr{B}_c(\theta)) \mid \theta \in \Theta\}$, where $\mathscr{B}_c(\theta) = \{(r,y,u) \mid r = \theta^T w(u,y)\}$ and θ is a 8×2 matrix of constant parameters in $\mathbb{R}$. For purposes of this simulation let "the true but unknown plant" be given by the following state space realisation of the plant given in ([3],[4],[5])

$$\begin{aligned} \dot{x} &= A_p x + B_p u \\ y &= C_p x \end{aligned}$$

where

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & -3 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & -5 & -6 & -4 \end{bmatrix},$$

$$B_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_p = \begin{bmatrix} 0 & 1 & 1 & -4 & -7 & -7 & -2 \\ 1 & 0 & 0 & 1 & 3 & 3 & 1 \end{bmatrix}.$$

Let the reference signal

$$r(t) = \begin{bmatrix} \sin(5t)+\sin(7t)+\sin(10t) \\ \sin(6t)+\sin(8t)+\sin(9t) \end{bmatrix}, \quad \forall t \geq 0.$$

as in ([3],[4],[5]). We obtain (y_{data}, u_{data}) by closing the loop with the initial controller associated to the matrix of parameters

$$\theta(0) = \begin{bmatrix} -4 & 0 \\ 0 & -1/2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

([3],[4],[5]) and the initial plant state given by $x = 0$. Thus we are able to use our theory to compute a new controller parameter $\theta(t)$ based on the data available at any given time $t = \tau$. Controller adaptation is achieved by repeating this operation periodically as time τ evolves and (y_{data}, u_{data}) accumulates, in order to update the controller parameter θ. Using this procedure to update the controller parameter vector $\theta(t)$ every 5 seconds, starting at time $\tau = 5$, we obtained the simulation results shown in figures 2 and 3, where $y_m = HM_0[r]$.

6 Concluding Remarks

The problem of fitting controllers to data was examined for one class of MIMO controllers used in model reference adaptive control ([3],[4],[5]). The formulation of the MIMO problem lead to an optimization problem where a best $2nm \times m$ matrix is looked for. The solution to this problem was shown to be reduced to the solution of a problem where a best $2nm^2 \times 1$ vector is looked for, which is solved in [7] when dealing with the identification of SISO controllers.Thus the problem was solved and an algorithm for fitting MIMO controllers to data was obtained. This algorithm was applied to a plant and reference model which appeared in the model reference adaptive control literature ([3],[4],[5]). Simulation results were provided which illustrated the applicability of the method. On the theoretical side, it was shown that the formulation of the problem of optimally fitting controllers to data illustrated an unifying link between the design of model reference adaptive control for SISO linear systems and the design of model reference adaptive control for MIMO linear systems.

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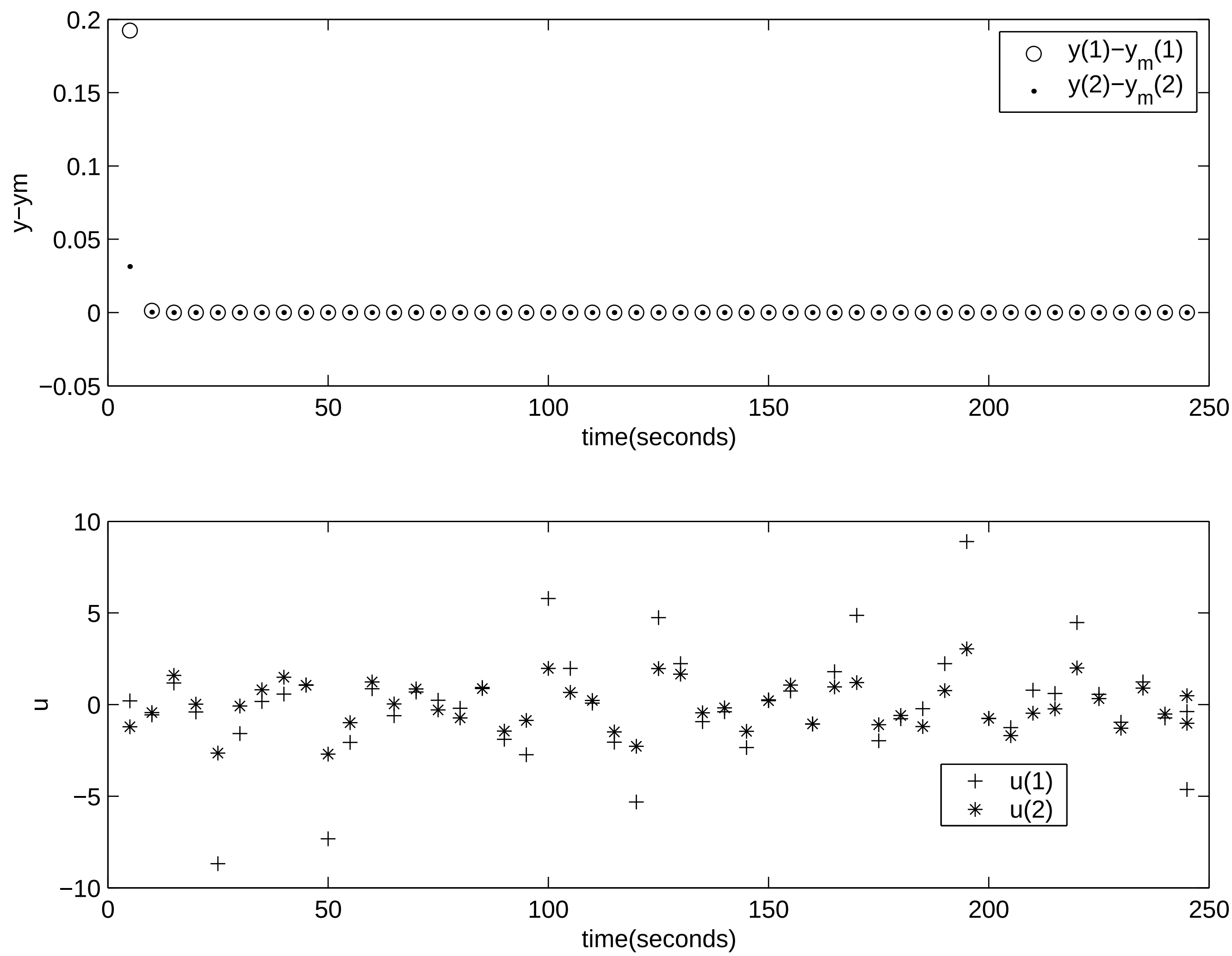

Figure 2: Simulation Results

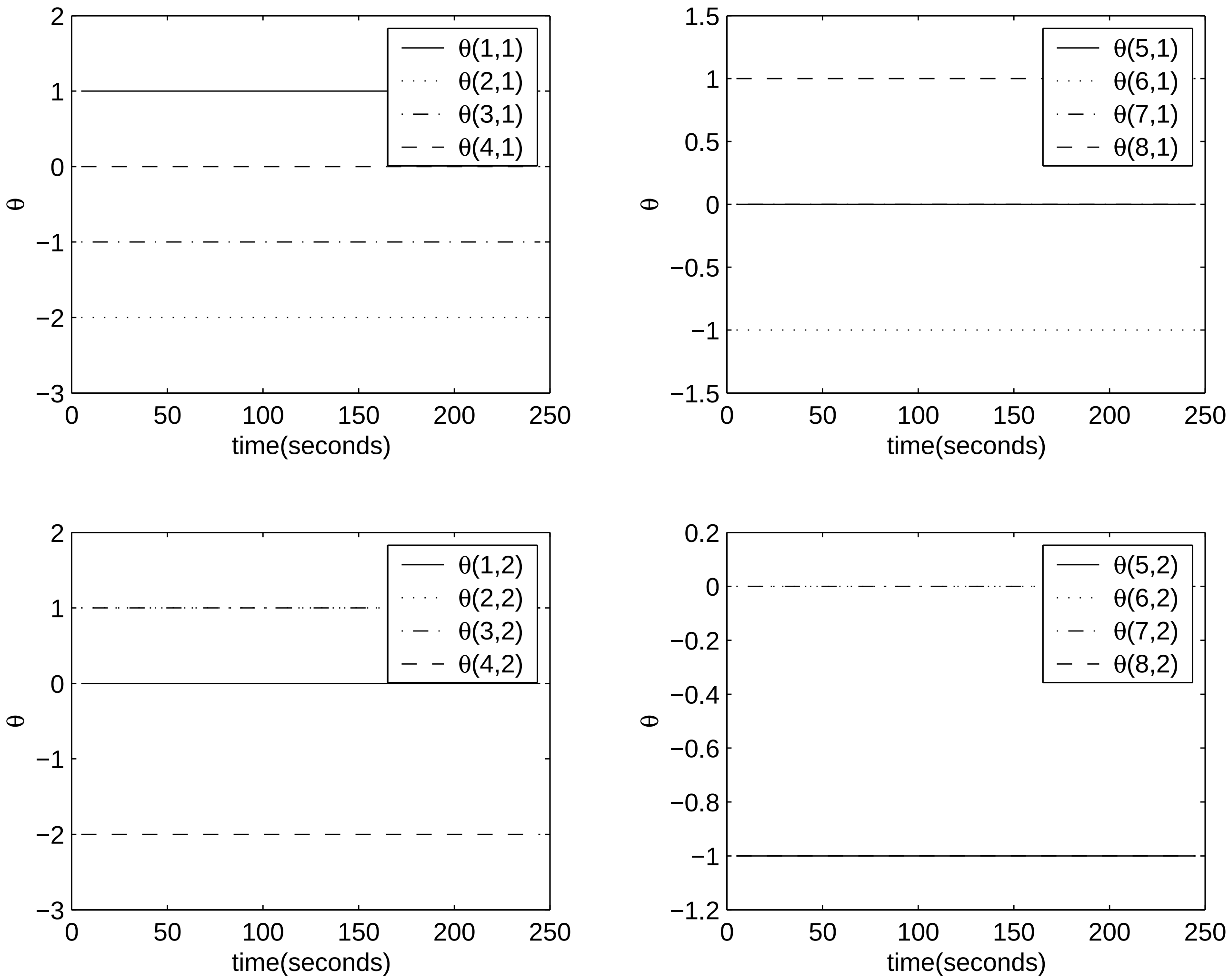

Figure 3: Simulation Results