Submitted June 2, 2003 to *Systems and Control Letters*. to appear 2005.

All Stability Multipliers for Repeated MIMO Nonlinearities*

Ricardo Mancera and Michael G. Safonov†

Abstract

For block structured monotone or incrementally positive n-dimensional nonlinearities, the largest class of convolution operators (stability multipliers) that preserve positivity is derived. These multipliers can be used in conjunction with positivity and IQC stability criteria to evaluate stability and robustness of MIMO feedback systems.

keywords: nonlinear stability multipliers positivity input-output robustness

1 INTRODUCTION

In this note we are interested in studying the input-output stability of feedback systems of the form as in Figure 1. By using the Passivity Theorem [2], we are able to conclude that this system is stable, provided the LTI stable plant $\mathbf{G}$ is strongly positive and the nonlinearity in the feedback path is positive. The main difficulty in applying this result, is that these positivity conditions usually do not hold immediately. The basic idea behind the multipliers technique is that by multiplying these operators (the plant $\mathbf{G}$ and the nonlinearity) by appropriately chosen 'multipliers' the product can be made to satisfy the conditions of the Passivity Theorem. More precisely, the role of linear time-invariant (LTI) stability multipliers in nonlinear stability theory is to permit less conservative stability conclusions. Such multipliers are useful in conjunction with positivity and related conic-sector and IQC-type stability criteria (cf. [7, 8, 10, 11, 14]). By optimizing over a class of positivity preserving multipliers for a given nonlinearity, one reduces the conservativeness of stability conclusions derived from positivity stability criteria. The key property of such multipliers $\mathcal{M}$ is that they preserve positivity of a monotone or incrementally positive nonlinearity $\mathbf{N}$, in the sense that $\mathbf{M} \in \mathcal{M}$ implies positivity of the operator $\mathbf{M}^*\mathbf{N}$, where $\mathbf{M}^*$ denotes the adjoint of $\mathbf{M}$. Zames and Falb [15, 16] studied continuous-time SISO feedback systems with a single monotone SISO nonlinearity and of the form as in Figure 1. Going beyond earlier multiplier results of Popov and others that had been derived via the Kalman-Yakubovich-Popov (KYP) lemma, they employed the input-output approach to derive a remarkably broad class of LTI stability multipliers that preserve positivity of SISO monotone nonlinearities which subsequently was proved to be the entire class of positivity preserving LTI stability multipliers for the continuous-time SISO case [4, 5]. All LTI stability multipliers for the discrete-time non-repeated SISO case were derived by Willems [13]. All stability multipliers for repeated SISO nonlinearities were found by Kulkarni and Safonov [4, 5], extending a result of D'Amato et al. [1].

*Research supported in part by AFOSR Grant F49620-01-1-0302.
†Authors are with the Department of Electrical Engineering at the University of Southern California, Los Angeles, CA 90089-2563. Email: msafonov@usc.edu.

Table 1: Relations between results characterizing all positivity-preserving stability multipliers.

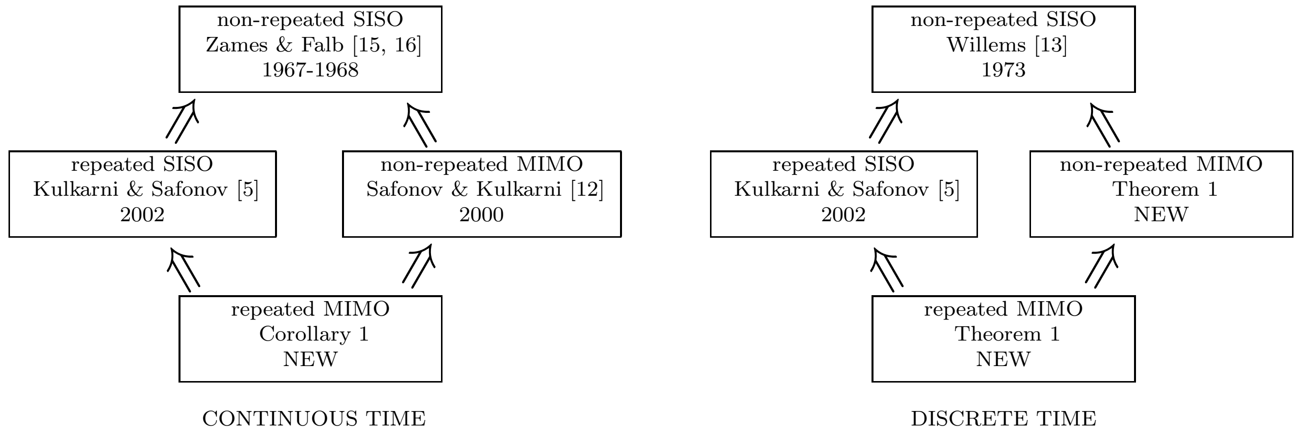

Table 1 summarizes the contributions of various researchers to the problem of characterizing all stability multipliers for various classes of nonlinearities, including our current results for the case of repeated MIMO nonlinearities. Note that the repeated MIMO results imply the earlier results, which follow as special cases of Theorem 1 and Corollary 1 of the present paper.

Algorithms for a practical usage of multipliers for SISO monotone nonlinearities case and repeated SISO nonlinearities exist (see [6] and references therein).

This note focuses on the characterization of the class of all multipliers that preserve positivity of a MIMO monotone or incrementally positive nonlinearity $\mathbf{N}$ that have a diagonally repeated block structure. These multipliers can be used to prove stability of MIMO discrete time feedback systems as illustrated in Figure 1 and where the following assumptions hold:

A1. x, y are n-dimensional discrete time real signals.

A2. $\mathbf{G} = [\mathbf{G}_{ij}]_{n\times n}$, is a discrete time LTI causal and stable operator.

A3. The diagonally repeated nonlinearity in the feedback loop is monotone or incrementally positive.

Continuous-time multiplier results are obtained by considering a discrete-time sampled system with vanishing sampling interval.

Systems with repeated monotone nonlinearities arise in a variety of engineering applications.

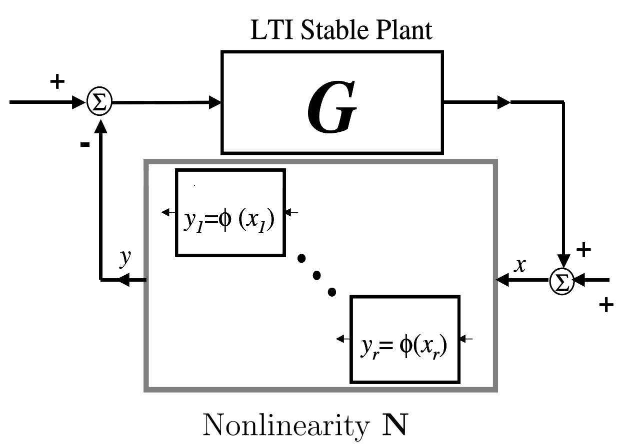

Figure 1: Repeated MIMO monotone or incrementally positive nonlinearity $\phi : \Re^s \to \Re^s$ in feedback loop.

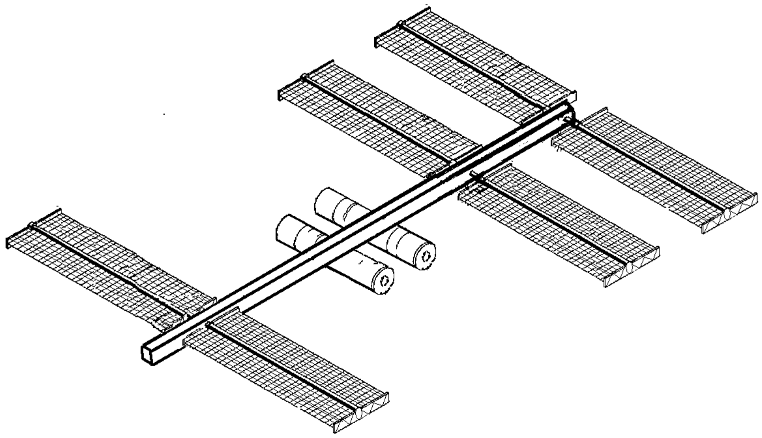

Figure 2: One place repeated MIMO monotone nonlinearities arise is in large space structures with self-similar substructures, like flexible solar panels that can bend or twist nonlinearly.

For example, in the SISO case we can mention artificial neural networks, communication networks and many other situations where they may appear as deadzones and actuator/sensor saturation nonlinearities. And, MIMO monotone nonlinearities may arise in nonlinear flexible structures (e.g., robot arms, space-structures). When such an elastic structure is twisted and/or compressed to position coordinate vector y by applying a collocated force vector x, the resultant potential energy associated with the elastic bending/twisting of the structure is $E(x) = \int_0^x \sigma' \rho(\sigma)\, d\sigma$ is a convex function of x; so, $\rho(x) = \nabla_x E(x)$ which means that $\rho : \Re^n \to \Re^n$ is a *monotone* MIMO nonlinearity (cf. Definition 2 below). Repeated monotone nonlinearities arise when an elastic mechanical structure includes two or more self-similar elastic substructures — see Fig. 2. Though such structures may be approximately linear for small disturbances, analysis of large amplitude motion requires nonlinear techniques.

The results are organized as follows. In Section 2, necessary terminology is introduced and background results are stated. The problems of concern are formally posed in Section 3. Preliminary lemmas are in Section 4, main results are presented in Section 5. Finally in section 6 main results are briefly discussed and conclusions summarized.

2 PRELIMINARIES

The notation used is summarized in Table 2.

A few definitions are in order before we can formally pose our problem. The definition of monotone mapping is motivated by the results in Safonov et al. [12], which suggest that this is the appropriate extension to n-dimensions of the concept of incrementally positive mapping introduced by Zames and Falb [15, 16]. At the end of the section, Lemma 1 presents a useful characterization of the monotone mappings.

Definition 1 (Incrementally positive mapping, [13])

Let ρ be a multivalued mapping from $Do(\rho) \subset \Re^n$ into $\Re^n$. Then ρ is said to be incrementally positive on $Do(\rho)$ if for all x, $\widehat{x} \in Do(\rho)$ one has

$$\langle x - \widehat{x}, y - \widehat{y} \rangle \geq 0$$

Table 2: Notation.

Symbol	Meaning
$\Re$	Set of real numbers
l_2	Space of infinite and real sequences $\mathbf{z} = \{z(k)\}$such that $\sum \lvert z(k)\rvert^2$ is summable
L_2	Space of real functions $z(t)$ such that $\int_{-\infty}^{\infty} \lvert z(t)\rvert^2$ is finite
l_2^n	Space of infinite sequences $\mathbf{z} = \{z(k)\}$ with $z(k) \in \Re^n$ and $\sum \lvert z(k)\rvert^2$ summable
$\mathcal{L}(\mathsf{A}, \mathsf{A})$	Space of bounded linear operators from A into A
I_n	$n \times n$ Identity matrix
$\text{herm}\{D\}$	$\frac{1}{2}\{D + \overline{D}'\}$ for $D \in \Re^{n\times n}$
$\text{diag}\{x_1, \ldots, x_n\}$	Diagonal matrix with $x_1, \ldots, x_n$ along its diagonal
$\otimes$	Kronecker product of matrices
$Do(\rho)$	Domain of the mapping ρ
$\langle x, y\rangle$	Inner product between x and y
MIMO	Multi-Input-Multi-Output
SISO	Single-Input-Single-Output
LTI	Linear Time Invariant

for all pairs (x, y) and $(\widehat{x}, \widehat{y})$ in the graph of ρ.

Definition 2 (Monotone mapping)
A multivalued mapping from $Do(\rho) \subset \Re^n$ into $\Re^n$ will be called monotone on $Do(\rho)$ if and only if one has $\rho(x) \subset \partial f(x)$ for every $x \in Do(\rho)$, where f is some closed proper convex function on $\Re^n$ and $\partial f(x)$ is the subdifferential of f at x.

Definition 3 (r-repeated MIMO mapping)
A multivalued mapping from $Do(\rho) \subset \Re^n$ into $\Re^n$ such that $0 \in \rho(0)$ and generated by r-fold repeated mapping $\rho = \text{diag}\,(\phi, \ldots, \phi)$ and $\phi : \Re^s \to \Re^s$ being a monotone (incrementally positive) mapping, $n = rs$ will be called r-repeated MIMO monotone (incrementally positive).

Definition 4 (Doubly hyperdominant matrix, [13])
A real $(p \times p)$ matrix $M = (m_{kl})$ is said to be doubly hyperdominant if

$$m_{kl} \leq 0 \text{ for } k \neq l \quad \text{and} \quad \sum_{k=1}^{p} m_{kl} \geq 0, \; \sum_{l=1}^{p} m_{kl} \geq 0 \quad \forall\; k, l$$

If, additionally,

$$\sum_{k=1}^{p} m_{kl} = \sum_{l=1}^{p} m_{kl} = 0 \quad \forall\; k, l$$

then M is said to be doubly hyperdominant with zero excess.

The definition of an operator $M \in \mathcal{L}(\mathsf{l}_2, \mathsf{l}_2)$ being doubly hyperdominant, is given in terms of its associated array $\{m_{kl}\}$ for integers k, l and is completely analogous to the one for the matrix M. The only difference is that the subindices k and l run from $-\infty$ to $+\infty$. In the continuous time case, these notions are defined in an analogous manner in terms of the kernel $m(t,\tau)$ of an operator $M \in \mathcal{L}(\mathsf{L}_2, \mathsf{L}_2)$ with integrals replacing sums.

Lemma 1

A multivalued mapping $\rho : Do(\rho) \subset \Re^n$ into $\Re^n$ is monotone if and only if for any permutation π of the integers, and for any sequences $\{x(k)\}, \{\widehat{x}(k)\}$ such that $x(k) \in Do(\rho),\ \widehat{x}(k) \in \rho(x(k))$, we have

$$\sum_k \langle x(k), \widehat{x}(k)\rangle \geq \sum_k \langle x(k), \widehat{x}(\pi(k))\rangle \tag{1}$$

Proof:

For any cyclic permutation of the form

$$\pi_c : (1, \ldots, q) \to (q, 1, 2, \ldots, q-2, q-1)$$

the result follows from [9] (Theorem 24.8) after a rearrangement of inequality (1). Now, any arbitrary permutation π, can written as a product $\pi = \prod \pi_i$ where the π_i are cyclic and disjoint permutations. Let C_i be the set of integers 'moved' by π_i then, in particular for $i \neq j$, $C_i \cap C_j = \emptyset$ and we can decompose

$$\sum_k \langle x(k), \widehat{x}(k)\rangle = \sum_i \sum_{k \in C_i} \langle x(k), \widehat{x}(k)\rangle$$

and by the remark about cyclic permutations,

$$\sum_{k \in C_i} \langle x(k), \widehat{x}(k)\rangle \geq \sum_{k \in C_i} \langle x(k), \widehat{x}(\pi_i(k))\rangle$$

for each i. Thus, from the previous two equations we get the result. □

Clearly ρ monotone implies ρ incrementally positive (take π that interchange only two elements). It can be shown, using a rearrangement inequality due to Hardy, Littlewood and Polya [3] (p. 277) that for $n = 1$, the incrementally positive mappings and the monotone mappings are the same. However, when $n > 1$ there exists incrementally positive mappings, which are not monotone. For example, when ρ is a (single valued) linear transformation from $\Re^n$ to $\Re^n$, ρ is monotone if and only if its matrix representation Q is symmetric and positive semidefinite (as may be deduced from the definition of monotone mapping). But ρ is incrementally positive if merely $\langle x(i) - x(j), Q(x(i) - x(j))\rangle \geq 0$ for all $x(i)$, $x(j)$, in other words if $\mathrm{herm}\{Q\}$ is positive semidefinite.

3 PROBLEM FORMULATION

Problem 1 (All Discrete-Time Stability Multipliers)

Find necessary and sufficient conditions on the convolution operator $\overline{M} \in \mathcal{L}(\mathsf{l}_2^n, \mathsf{l}_2^n)$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all sequences x, $y \in \mathsf{l}_2^n$ such that $y(k) \in \rho(x(k))$ and ρ is a r-repeated MIMO monotone (incrementally positive, $n > 1$) mapping.

Problem 2 (All Continuous-Time Stability Multipliers)

Find necessary and sufficient conditions on the convolution operator $\overline{M} \in \mathcal{L}(\mathsf{L}_2^n, \mathsf{L}_2^n)$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all x, $y \in \mathsf{L}_2^n$ such that $y(t) \in \rho(x(t))$ and ρ is a r-repeated MIMO monotone (incrementally positive, $n > 1$) mapping.

4 PRELIMINARY RESULTS

In this section, several results of [13] stated in terms of similarly ordered finite and infinite sequences of real numbers are extended to the case where we have finite and infinite sequences of vectors in $\Re^n$ and the concept of similarly ordered sequence has been replaced by the one of incrementally positive or monotone mapping. The goal is to characterize the set of LTI operators $\overline{M}$ such that $\left\langle x, \overline{M}y \right\rangle \geq 0$ for all sequences x, y in the graph of a incrementally positive or monotone mapping.

To simplify notation, in the following $\mathbf{0}$, $\mathbf{1}$ and $-\mathbf{1}$ will denote n-dimensional vectors of zeros, ones and negative ones.

Lemma 2 (Generalization of Willems [13, Thm. 3.7])

A necessary and sufficient condition on the $(np \times np)$ matrix $\overline{M}$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all finite sequences $x = \{x(1), \ldots, x(p)\}$, $y = \{y(1), \ldots, y(p)\}$, such that $x(k) \in \Re^n$, $y(k) \in \rho(x(k))$, and ρ is a monotone mapping is that $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} \overline{M}_{11} & \cdots & \overline{M}_{1p} \\ \vdots & \ddots & \vdots \\ \overline{M}_{p1} & \cdots & \overline{M}_{pp} \end{bmatrix} = \begin{bmatrix} m_{11}I_n & \cdots & m_{1p}I_n \\ \vdots & \ddots & \vdots \\ m_{p1}I_n & \cdots & m_{pp}I_n \end{bmatrix}$$

where $M = (m_{kl})$ is a $(p \times p)$ doubly hyperdominant matrix with zero excess.

Proof of sufficiency:

Let $\overline{M}$ be of the claimed form. Let r be any positive number such that $r \geq m_{kl}$ for all k, l. Clearly $M = r\left[I_p - \frac{1}{r}(rI_p - M)\right]$. Now, since $\frac{1}{r}(rI_p - M)$ is a doubly stochastic matrix, it can be decomposed as $\sum_{i=1}^{N} \alpha_i P_i$ with $\alpha_i \geq 0$, $\sum_{i=1}^{N} \alpha_i = 1$, where P_i is a $(p \times p)$ permutation matrix. Thus M can be written as $M = \sum_{i=1}^{N} \beta_i (I_p - P_i)$ with $\beta_i \geq 0$, $\beta_i = r\alpha_i$. Therefore, $\overline{M} = M \otimes I_n = \sum_{i=1}^{N} \beta_i (I_p - P_i) \otimes I_n = \sum_{i=1}^{N} \beta_i (I_{n \cdot p} - \overline{P}_i)$, where $\overline{P}_i = P_i \otimes I_n$. This decomposition of $\overline{M}$ shows that it is enough to prove the claim for the matrices $\left(I_{n \cdot p} - \overline{P}_i\right)$. This is precisely what is stated in inequality (1).

Proof of necessity:

The proof is done in several steps.

First, let us assume that $\overline{M}$ is of the form $\overline{M} = M \otimes I_n$, but the matrix M fails to be doubly hyperdominant with zero excess because $m_{ij} > 0$ for some $i \neq j$. The mapping ρ from $x = \{\mathbf{0}, \ldots, \mathbf{0}, \mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}\}$ into $y = \{\mathbf{0}, \ldots, \mathbf{0}, -\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}\}$ such that for each k, the vector $x(k)$ maps into the vector $y(k)$, is monotone, as can be seen from Lemma 1, also the $\mathbf{1}$ and $-\mathbf{1}$ are in the i-th and j-th positions $(i \neq j)$ of x and y respectively. Furthermore

$$\left\langle x, \overline{M}y \right\rangle = \langle \mathbf{1}, m_{ij} I_n (-\mathbf{1}) \rangle = -n m_{ij} < 0$$

Assume next, that $\overline{M}$ is of the form $\overline{M} = M \otimes I_n$, but the matrix M fails to be doubly hyperdominant with zero excess, because $\sum_{k=1}^{p} m_{kj} \neq 0$ for some j. The mapping ρ from $x = \{\mathbf{1}, \ldots, \mathbf{1}, (1+\varepsilon)\mathbf{1}, \mathbf{1}, \ldots, \mathbf{1}\}$ into $y = \left\{\mathbf{0}, \ldots, \mathbf{0}, \frac{1}{\varepsilon}\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}\right\}$ with $\varepsilon \neq 0$ such that each vector $x(k)$ maps into the vector $y(k)$,for all k, is monotone, as can be seen from Lemma 1, and the elements $(1+\varepsilon)\mathbf{1}$ and $\frac{1}{\varepsilon}\mathbf{1}$ are in the j-th place of x and y respectively. This leads to $\left\langle x, \overline{M}y \right\rangle =$

$n(\frac{1}{\varepsilon}\sum_{k=1}^{p} m_{kj} + m_{jj})$. Thus, by taking ε sufficiently small and of appropriate sign $\left\langle x, \overline{M}y \right\rangle$ can then indeed be made negative.

Assume next, that $\overline{M}$ is not of the form $\overline{M} = M \otimes I_n$ because $\overline{M}_{kl}$ is not diagonal for some k, l. Assume then that the (i, j) (for $i \neq j$) entry of the submatrix $\overline{M}_{kl}$ (denoted by $(\overline{M}_{kl})_{ij}$) is not zero. The mapping ρ from $x = \{\mathbf{0}, \ldots, \mathbf{0}, A, \mathbf{0}, \ldots, \mathbf{0}\}$ into $y = \{\mathbf{0}, \ldots, \mathbf{0}, B, \mathbf{0}, \ldots, \mathbf{0}\}$ such that for each k, the vector $x(k)$ maps into the vector $y(k)$, is monotone, as can be seen from Lemma 1, with A and B in the k-th and l-th positions, where $A = (0, \ldots, 0, 1, 0, \ldots, 0) \in \Re^n$ and $B = (0, \ldots, 0, \delta, 0, \ldots, 0) \in \Re^n$ with 1 and δ in the i-th and j-th positions, respectively. Therefore,

$$\left\langle x, \overline{M}y \right\rangle = \left\langle A, \overline{M}_{kl}B \right\rangle = \delta(\overline{M}_{kl})_{ij}$$

can be made negative by choosing δ of the appropriate sign.

Finally, assume that $\overline{M}$ is not of the form $\overline{M} = M \otimes I_n$ because $\overline{M}_{kl}$ is diagonal, but not of the form $m_{kl}I_n$, let us say $(\overline{M}_{kl})_{ii} > (\overline{M}_{kl})_{jj}$ for some $i \neq j$ then, as before, we can check that the mapping ρ such that each vector $x(k)$ maps into the vector $y(k)$,for all k, is monotone, as can be seen from Lemma 1, where

$$x = \{\mathbf{0}, \ldots, \mathbf{0}, C, \mathbf{0}, \ldots, \mathbf{0}\}, \quad y = \{\mathbf{0}, \ldots, \mathbf{0}, D, \mathbf{0}, \ldots, \mathbf{0}\}$$

with C and D in the k-th and l-th positions, and

$$C = (0, \ldots, 0, -1, 0, \ldots, 0, 1, 0, \ldots, 0) \in \Re^n, \quad D = (1, \ldots, 1) \in \Re^n$$

with C having -1 and 1 in the i-th and j-th positions, thus

$$\left\langle x, \overline{M}y \right\rangle = \left\langle C, \overline{M}_{kl}D \right\rangle = (\overline{M}_{kl})_{jj} - (\overline{M}_{kl})_{ii} < 0$$

A similar argument works for the case when $(\overline{M}_{kl})_{ii} < (\overline{M}_{kl})_{jj}$. □

For clarity we state separately the following very closely related result to Lemma 2.

Lemma 3 [Incrementally positive case]

A necessary and sufficient condition on the $(np \times np)$ matrix $\overline{M}$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all finite sequences $x = \{x(1), \ldots, x(p)\}$, $y = \{y(1), \ldots, y(p)\}$, such that $x(k) \in \Re^n$, $y(i) \in \rho(x(k))$, and ρ is an incrementally positive $(n > 1)$ mapping is that $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} \overline{M}_{11} & \cdots & \overline{M}_{1p} \\ \vdots & \ddots & \vdots \\ \overline{M}_{p1} & \cdots & \overline{M}_{pp} \end{bmatrix} = \begin{bmatrix} m_{11}I_n & \cdots & m_{1p}I_n \\ \vdots & \ddots & \vdots \\ m_{p1}I_n & \cdots & m_{pp}I_n \end{bmatrix}$$

where $M = (m_{kl})$ is a $(p \times p)$ doubly hyperdominant and symmetric matrix with zero excess.

Proof of sufficiency:

Let M be doubly hyperdominant with zero excess, then, from the proof of sufficiency of Lemma 2 we conclude that M is of the form $M = \sum_{i=1}^{N} \beta_i(I_n - P_i)$, for some permutation matrices P_i and some $\beta_i > 0$. The fact that M is also symmetric implies that we can rewrite M as $M = \sum_{j=1}^{N'} \beta'_j(I_n - P^s_j)$

for some symmetric permutation matrices P_j^s and some $\beta_j' > 0$. This decomposition of M implies that $\overline{M} = \sum\limits_{j=1}^{N'} \beta_j'(I_{n\cdot p} - \overline{P}_j^s)$, where $\overline{P}_j^s = P_j^s \otimes I_n$, therefore, it is enough to prove the claim for the matrices $\left(I_{n\cdot p} - \overline{P}_j^s\right)$. This follows directly from the symmetry of $\overline{P}_j^s$ and the fact that ρ is incrementally positive.

Proof of necessity:

The fact that $\overline{M}$ must be of the from $\overline{M} = M \otimes I_n$, where M is doubly hyperdominant with zero excess follows immediately from Lemma 2: If $\left\langle x, \overline{M}y \right\rangle$ is positive for all sequences in the graph of ρ incrementally positive, then as monotone implies incrementally positive, it is in particular positive for all sequences in the graph of monotone maps. Finally, assume that $M = \sum\limits_{i=1}^{N} \beta_i(I_n - P_i)$ but it is not symmetric. It is straightforward to show that M can be decomposed as

$$M \equiv M_1 + M_2 \equiv \sum\nolimits_{i=1}^{N_1} \beta_i^s(I - P_i^s) + \sum\nolimits_{j=1}^{N_2} \beta_j^{ns}(I - P_j^{ns})$$
$$\text{with } \; N_1 + N_2 = N, \; N_2 \geq 1$$

where M_1, M_2 are doubly hyperdominant with zero excess, M_1 is symmetric, M_2 is not symmetric and $(M_2)_{ij} = 0$ or $(M_2)_{ji} = 0$ for $j \neq i$. This implies in particular that the permutations P_i^s and P_j^{ns} can be chosen to be symmetric and not symmetric matrices respectively. Choose any P_j^{ns}, say $P_{\hat{j}}^{ns}$ such that $(P_{\hat{j}}^{ns})_{lq}$ is one, for some $l \neq q$. We can verify that the mapping ρ such that for each k, the vector $x(k)$ maps into the vector $y(k)$ is incrementally positive (but not monotone), where $x = \{\mathbf{0}, \ldots, \mathbf{0}, A, \mathbf{0}, \ldots, \mathbf{0}, B, \mathbf{0}, \ldots, \mathbf{0}\}$ and $y = \{\mathbf{0}, \ldots, \mathbf{0}, -B, , \mathbf{0}, \ldots, \mathbf{0}, A, \mathbf{0}, \ldots, \mathbf{0}\}$, $A = (1, , \ldots, 0) \in \Re^n$, $B = (0, 1, 0, \ldots, 0) \in \Re^n$ with A, B in the l^{th} and q^{th} component of x and $-B$, A in the l^{th} and q^{th} component of y respectively. Furthermore

$$\left\langle x, \overline{M}y \right\rangle =$$

$$= \left\langle x, (\sum\nolimits_{i=1}^{N_1} \beta_i^s(I - \overline{P}_i^s) + \sum\nolimits_{j=1}^{N_2} \beta_j^{ns}(I - \overline{P}_j^{ns}))y \right\rangle =$$

$$= \left\langle x, \sum\nolimits_{j=1}^{N_2} \beta_j^{ns}(I - \overline{P}_j^{ns})y \right\rangle$$

Also notice that there is no j such that the permutation $(P_j^{ns})_{ql} = 1$, because $(M_2)_{ij} = 0$ or $(M_2)_{ji} = 0$ for $j \neq i$. Thus

$$\left\langle x, \sum\nolimits_{j=1}^{N_2} \beta_j^{ns}(I - \overline{P}_j^{ns})y \right\rangle \leq$$

$$\leq \beta_{\hat{j}}^{ns} \left\langle x, (I - \overline{P}_{\hat{j}}^{ns})y \right\rangle = -\beta_{\hat{j}}^{ns} < 0$$

Therefore, M must be symmetric. □

Remark

By adding the symmetry condition to M we can substitute the word monotone by incrementally positive, as illustrated in Lemmas 2, 3. This also holds for all the lemmas and theorems that follow. For the sake of brevity we will combine these results in one statement.

Under the additional assumption that $\mathbf{0} \in \rho(\mathbf{0})$, we can eliminate the zero excess condition in Lemmas 2, 3.

Lemma 4 (Generalization of Willems [13, Thm. 3.8])

A necessary and sufficient condition on the $(np \times np)$ matrix $\overline{M}$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all finite sequences $x = \{x(1), \ldots, x(p)\}$, $y = \{y(1), \ldots, y(p)\}$,such that $x(k) \in \Re^n$, $y(k) \in \rho(x(k))$ and $\mathbf{0} \in \rho(\mathbf{0})$ where ρ is a monotone (incrementally positive, $n > 1$) mapping is that $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} M_{11} & \cdots & M_{1p} \\ \vdots & \ddots & \vdots \\ M_{p1} & \cdots & M_{pp} \end{bmatrix} = \begin{bmatrix} m_{11}I_n & \cdots & m_{1p}I_n \\ \vdots & \ddots & \vdots \\ m_{p1}I_n & \cdots & m_{pp}I_n \end{bmatrix}$$

where $M = (m_{kl})$ is a $(p \times p)$ doubly hyperdominant (doubly hyperdominant and symmetric) matrix.

Proof of sufficiency:

Let M_A be the $(p+1 \times p+1)$ matrix, obtained via augmenting M as follows $m_{k,p+1} = -\sum_{l=1}^{p} m_{kl}$, $m_{p+1,l} = -\sum_{k=1}^{p} m_{kl}$ for $k, l \leq p$, and $m_{p+1,p+1} = \sum_{k=1,l}^{p} m_{kl}$. Then defining also the augmented finite sequences $x_A = \{x, \mathbf{0}\}$, $y_A = \{y, \mathbf{0}\}$ it follows from Lemma 2 (3) that

$$\left\langle x, \overline{M}y \right\rangle = \left\langle x_A, \overline{M}_A y_A \right\rangle \geq 0$$

Proof of necessity:

The same sequences used in the necessity proof of Lemma 2 (3) can be used here to obtain the result. $\square$

The following extension to Lemma 4 holds.

Lemma 5 (Generalization of Willems [13, Thm. 3.11])

A necessary and sufficient condition on the operator $\overline{M} \in \mathcal{L}(\mathsf{l}_2^n, \mathsf{l}_2^n)$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all infinite sequences $x = \{x(k)\}$, $y = \{y(k)\}$, such that $x(k) \in \Re^n$, $y(k) \in \rho(x(k))$ and $\mathbf{0} \in \rho(\mathbf{0})$ where ρ is a monotone (incrementally positive, $n > 1$) mapping is that $\overline{M}$ be of the form $\overline{M} = M \otimes I_n$, that is

$$\overline{M} = \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & M_{11} & \cdots & M_{1p} & \cdots \\ & \vdots & & \vdots & \\ \cdots & M_{p1} & \cdots & M_{pp} & \cdots \\ & \vdots & & \vdots & \end{bmatrix} =$$

$$= \begin{bmatrix} & \vdots & & \vdots & \\ \cdots & m_{11}I_n & \cdots & m_{1p}I_n & \cdots \\ & \vdots & & \vdots & \\ \cdots & m_{p1}I_n & \cdots & m_{pp}I_n & \cdots \\ & \vdots & & \vdots & \end{bmatrix}$$

where $M = (m_{kl})$ is doubly hyperdominant (doubly hyperdominant and symmetric).

Proof:

By the previous Lemma 4, all finite truncations of the infinite sum in the inner product $\left\langle x, \overline{M}y \right\rangle$ yield a nonnegative number. Therefore the limit, since it exists, is also nonnegative. $\square$

5 MAIN RESULTS

In the particular case that $\overline{M}$ is a convolution operator and the nonlinearity is r-repeated MIMO, the following two theorems provide answers to Problems 1 and 2, respectively. The following theorem generalizes of a non-repeated SISO nonlinearity result of Willems [13, Thm. 3.12].

Theorem 1 (All Stability Multipliers for Repeated MIMO Nonlinearity—Discrete Time)

A necessary and sufficient condition on the convolution operator $\overline{M} \in \mathcal{L}(\mathsf{l}_2^n, \mathsf{l}_2^n)$ for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all sequences x, $y \in \mathsf{l}_2^n$ such that $y(k) \in \rho(x(k))$ and ρ is a r-repeated MIMO monotone (incrementally positive, $n > 1$) mapping is that $\overline{M}$ be of the form

$$\overline{M} = \{M_{kl}\} = \{M_{k-l}\}$$

where

$$M_{kl} = \begin{bmatrix} m_{11}^{k-l} I_s & \cdots & m_{1r}^{k-l} I_s \\ \vdots & \ddots & \vdots \\ m_{r1}^{k-l} I_s & \cdots & m_{rr}^{k-l} I_s \end{bmatrix} \tag{2}$$

and the array $\{M_{kl}\}$ is doubly hyperdominant (doubly hyperdominant and symmetric).

Proof:

The condition that $\overline{M}$ is a convolution operator in $\mathcal{L}(\mathsf{l}_2^n, \mathsf{l}_2^n)$ implies that $\{M_{kl}\}$ is block Toeplitz. The structure of M_{kl} as in (2) and its doubly hyperdominance (and symmetry) follow from Lemma 5. $\square$

Remark

Theorem 1 gives a necessary and sufficient MIMO generalization (discrete time) of the sufficient SISO stability conditions derived by Zames-Falb [16, Thm. 2] (continuous time). The sufficiency conditions in [15, 16] were generalized to the MIMO case by Safonov-Kulkarni [12], Theorem 1, (continuous time) again as sufficient conditions only.

Corollary 1 (All Stability Multipliers for Repeated MIMO Nonlinearity—Continuous Time)

Let $\overline{M} \in \mathcal{L}(\mathsf{L}_2^n, \mathsf{L}_2^n)$ be a convolution operator having impulse response

$$\widehat{M}(t) = M_0 \delta(t) - H(t) \tag{3}$$

with $H(t) \in \mathsf{L}_2^{n \times n}$ is continuous, and $M_0 \in \Re^{n \times n}$.

Then a necessary and sufficient condition for the inner product $\left\langle x, \overline{M}y \right\rangle$ to be nonnegative for all continuous signals $x \in \mathsf{L}_2^n$, $y \in \mathsf{L}_2^n$, $y(t) = \rho\{x(t)\} \in \Re^n$, and ρ is a r-repeated MIMO monotone (incrementally positive, $n > 1$) mapping is that M_0 be of the form

$$M_0 = \mathrm{diag}(m_1 I_s, \ldots, m_r I_s)$$

and

$$H(t) = \begin{bmatrix} h_{11}(t)I_s & \cdots & h_{1r}(t)I_s \\ \vdots & \ddots & \vdots \\ h_{r1}(t)I_s & \cdots & h_{rr}(t)I_s \end{bmatrix}$$

with $h_{ij}(t) \geq 0$ for all $i, j = 1, \ldots, r$ and that the matrix

$$M_0 - \int_{-\infty}^{\infty} H(t)\, dt \in \Re^{n \times n} \tag{4}$$

be doubly hyperdominant (doubly hyperdominant and symmetric).

Proof:

We shall discretize the continuous signals $x(t), y(t)$ and the operator $\overline{M} \in \mathcal{L}(\mathsf{L}_2^n, \mathsf{L}_2^n)$ by sampling the signals x, y with sampling interval ϵ. Let $x_{d,\epsilon}(k) = \sqrt{\epsilon} x_{c,\epsilon}(k\epsilon)$, $y_{d,\epsilon}(k) = \sqrt{\epsilon} y_{c,\epsilon}(k\epsilon)$. Then

$$\begin{aligned} \left\langle x, \overline{M} y \right\rangle &= \int_{-\infty}^{\infty} x'(t) \int_{-\infty}^{\infty} \widehat{M}(t-\tau) y(\tau)\, d\tau\, dt \\ &= \lim_{\epsilon \to 0} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} x_{d,\epsilon}(k) M_{kl,\epsilon} y_{d,\epsilon}(l) \end{aligned}$$

where

$$M_{kl,\epsilon} = \begin{cases} M_0 - H(0) & \text{if } k = l \\ -H(\epsilon(k-l)), & \text{otherwise.} \end{cases}$$

The condition (4) ensures that for ϵ small enough, $\{M_{kl,\epsilon}\}$ is doubly hyperdominant (doubly hyperdominant and symmetric). The result follows immediately from Theorem 1. $\square$

Remark The main contributions of this note are presented in Table 1. It is perhaps noteworthy that all the results shown in this table are in fact special cases of our main results Theorem 1 and Corollary 1.

6 CONCLUSIONS

The problem of identifying an expanded set of stability multipliers for two classes of repeated MIMO nonlinearities, monotone and incrementally positive nonlinear mappings has been examined. For each of these classes, the entire set of LTI multiplier matrices that preserve positivity has been characterized. Both the discrete time and continuous time cases have been treated, and multipliers previously known for more restrictive classes of nonlinearities have been seen to emerge as special cases. The results enable one to obtain less conservative stability robustness results for complex systems with several identical nonlinear elements, as for example elastic mechanical systems having two or more identical nonlinear subsystems (e.g., space structure truss elements or solar panels, robot arms, and so forth).

7 ACKNOWLEDGMENTS

We would like to thank Dr. Vishwesh Kulkarni for helpful comments related to this paper.

References

[1] F. D'Amato, A. Megretski, U. Jönsson, and M. Rotea. New results for analysis of systems with repeated nonlinearities. *Automatica*, 37(5):739–747, May 2001.

[2] C. A. Desoer and M. Vidyasagar. *Feeback Systems: Input-Output Properties*. Academic Press, New York, 1975.

[3] G. H. Hardy, T. E. Littlewood, and G. Polya. *Inequalities (second edition)*. Cambridge University Press, Cambrige, England, 1952.

[4] V. V. Kulkarni and M. G. Safonov. Multipliers for repeated MIMO nonlinearities. In *Proc. American Control Conf.*, pages 2292–2296, Arlington, VA, June 27–29, 2001.

[5] V. V. Kulkarni and M. G. Safonov. All multipliers for repeated monotone nonlinearities. *IEEE Trans. Autom. Control*, 47(7):1209–1212, July 2002.

[6] R. Mancera and M. G. Safonov. An algorithm to compute multipliers for repeated monotone nonlinearities. In *Proc. IEEE Conf. on Decision and Control*, pages 4162–4165, Las Vegas, NV, December 10-13, 2002. IEEE Press, New York.

[7] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans. Autom. Control*, AC-42(6):819–830, June 1997.

[8] A. N. Michel and R. K. Miller. *Qualitative Analysis of Large Scale Synamical Systems*. Academic Press, New York, 1977.

[9] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, New Jersey, 1970.

[10] M. G. Safonov. *Stability and Robustness of Multivariable Feedback Systems*. MIT Press, Cambridge, MA, 1980. Based on author's PhD Thesis, "Robustness and Stability Aspects of Stochastic Multivariable Feedback System Design," MIT, 1977.

[11] M. G. Safonov. Robust control, stability margin. In P. M. Pardalos and C. A. Floudas, editors, *Encyclopedia of Optimization*, volume 5, pages 44–49. Kluwer, Boston, MA, June 2001.

[12] M. G. Safonov and V. V. Kulkarni. Zames-Falb multipliers for MIMO nonlinearities. *Int J. Robust and Nonlinear Control*, 10(11/12):1025–1038, September - October 2000. George Zames Commemorative Issue (eds. K. Glover and M. G. Safonov).

[13] J. C. Willems. *The Analysis of Feedback Systems*. MIT Press, Cambridge, MA, 1973.

[14] G. Zames. On the input–output stability of time-varying nonlinear feedback systems — Parts I and II. *IEEE Trans. Autom. Control*, AC-11(2 and 3):228–238 and 465–467, April and July 1966.

[15] G. Zames and P. L. Falb. On the stability of systems with monotone and odd monotone nonlinearities. *IEEE Trans. Autom. Control*, AC-12(2):221–223, April 1967.

[16] G. Zames and P. L. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM J. Control and Optimization*, 6(1):89–108, 1968.