

Necessary and Sufficient Conditions for Stability of a Class of Second Order Switched Systems

Ayanendu Paul, Mehmet Akar, Michael G. Safonov and Urbashi Mitra

Abstract—For a special class of systems, it is shown that the existence of a Common Quadratic Lyapunov Function (CQLF) is necessary and sufficient for the stability of an associated switched system under arbitrary switching. Furthermore, it is shown that the existence of a CQLF for N ($N > 2$) subsystems is equivalent to the existence of a CQLF for every pair of subsystems. An algorithm is proposed to compute a CQLF for the subsystems, when it exists, using the left and right eigenvectors of a critical matrix obtained from a matrix pencil.

Index Terms—switched systems, stability, common quadratic Lyapunov function, M -matrix

I. PROBLEM STATEMENT

Consider the switched system

$$\Sigma_s : \dot{x}(t) = A(t)x(t), A(t) \in \mathcal{A} = \{A_1, A_2, \dots, A_N\} \quad (1)$$

where $x(t) \in \mathbb{R}^2$ is the state, and $A_i \in \mathbb{R}^{2 \times 2}$, $i = 1, 2, \dots, N$ are the system matrices for the subsystems

$$\Sigma_i : \dot{x}(t) = A_i x(t), i = 1, 2, \dots, N. \quad (2)$$

Throughout the paper, the negative of each matrix A_i (i.e., $-A_i$) is assumed to be an M -matrix¹. Therefore, each matrix $A = [a_{ij}]$ in the set \mathcal{A} satisfies $a_{ii} < 0$, $i = 1, 2$ and $a_{ij} \geq 0$, and is Hurwitz. The objective of this paper is to derive necessary and sufficient conditions for the stability of the switched system (1) under arbitrary switching between the system matrices $A_i, i = 1, 2, \dots, N$.

Clearly, if a common quadratic Lyapunov function (CQLF) exists for the subsystems $\Sigma_i, i = 1, 2, \dots, N$, then the switched system (1) is stable under arbitrary switching. The converse of this statement is not true in general [3], [4]. However, in this paper, we prove that the converse is indeed true for a specific class of systems.

The following notation will be used in sequel. Let $T = [t_{ij}] \in \mathbb{R}^{n \times m}$. T is said to be a non-negative matrix, and denoted as $T \succeq 0$ if $t_{ij} \geq 0$ for $1 \leq i \leq n, 1 \leq j \leq m$. Similarly, for two matrices $T_1, T_2 \in \mathbb{R}^{n \times m}$, we write $T_1 \succeq T_2$ if $T_1 - T_2 \succeq 0$. For $Q \in \mathbb{R}^{n \times n}$, $Q > 0$ denotes that Q is positive definite.

II. MAIN RESULTS

We first consider the two subsystem case.²

Proposition 1: For the systems in (1) and (2) with $N = 2$, the following statements are equivalent.

The authors are with the Department of Electrical Engineering-Systems, University of Southern California, Los Angeles, CA, 90089-2565, USA. E-mail: {ayanendp, akar, msafonov, ubli}@usc.edu

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¹See [1], [2] for the definition and properties of M -matrices.

²The proofs of the results in this section are relegated to the Appendix.

- (i) The switched system Σ_s is stable under arbitrary switching.
- (ii) The matrix pencil $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz for all $\alpha \in [0, 1]$.
- (iii) A CQLF exists for the subsystems Σ_1 and Σ_2 .

Remark 1: (i) In general, the existence of a CQLF is not necessary for the stability of a switched system under arbitrary switching [3], [4]. However, for the special class of systems under consideration, Proposition 1 states that stability of the switched system in (1) is equivalent to the existence of a CQLF for the subsystems in (2).

(ii) The stability properties of the switched system in (1) can be determined by checking the stability of the matrix pencil, $\alpha A_1 + (1 - \alpha)A_2$, $\alpha \in [0, 1]$, which can further be reduced to checking whether A_1 is Hurwitz and whether the matrix product $A_1^{-1}A_2$ has any negative eigenvalues [5]. Furthermore, the critical value of α for which $\alpha A_1 + (1 - \alpha)A_2$ has the largest real eigenvalue can be found easily [4].

Lemma 1: If the diagonal entries of the matrices for the systems in (1) and (2) with $N = 2$ are equal to -1 , then the following statements are equivalent.

- (i) The switched system Σ_s is stable under arbitrary switching.
- (ii) The matrix pencil $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz for all $\alpha \in [0, 1]$.
- (iii) A *diagonal* CQLF exists for subsystems Σ_1 and Σ_2 .

Remark 2: This type of system, whose system matrices have -1 on the diagonals, are widely encountered in problems of power control for wireless networks [6], [7].

The results of Lemma 1 can be extended as follows.

Theorem 1: If the diagonal entries of the matrices for the systems in (1) and (2) with $N > 2$ are equal to -1 , then the following statements are equivalent.

- (i) The switched system Σ_s is stable under arbitrary switching.
- (ii) All matrices in the convex hull $\sum_{i=1}^N \alpha_i A_i$ are Hurwitz for $\alpha_i \geq 0$, $i = 1, 2, \dots, N$, and $\sum_{j=1}^N \alpha_j = 1$.
- (iii) The matrix pencils $\alpha A_i + (1 - \alpha)A_j$ are Hurwitz for all $\alpha \in [0, 1]$, and all $i, j = 1, 2, \dots, N, i \neq j$.
- (iv) A *diagonal* CQLF exists for every pair of subsystems Σ_i and $\Sigma_j, i, j = 1, 2, \dots, N, i \neq j$.
- (v) A *diagonal* CQLF exists for the subsystems $\Sigma_i, i = 1, 2, \dots, N$ and it can be computed using the following algorithm.

Algorithm 1: (a) Among all pairs of the matrices A_i and $A_j, i, j = 1, 2, \dots, N, i < j$, determine (using Lemma 1) the matrix $A = \alpha A_i + (1 - \alpha)A_j$ which has the largest real eigenvalue for some $\alpha \in [0, 1]$.

(b) Solve for $v = [v_1, v_2]^T$ from $Av = \lambda v$.

(c) Compute the diagonal common Lyapunov matrix D as $D = \text{diag}[v_2/v_1, v_1/v_2]$. \square

Remark 3: In general, for $N > 2$, the existence of a CQLF for every pair of subsystems is necessary but not sufficient for the existence of a CQLF for all of the subsystems [4], [8]. However, the above theorem states that this condition is also sufficient for the special class of systems under consideration.

APPENDIX

A. Proof of Proposition 1

(i) \Rightarrow (ii): Suppose that the matrix pencil $\alpha A_1 + (1 - \alpha)A_2$ is unstable for some $\alpha = \alpha_c \in [0, 1]$. Then $A_{eq} = \alpha_c A_1 + (1 - \alpha_c)A_2$ is unstable which means that alternating switching between systems Σ_1 and Σ_2 utilizing Σ_1 for $\alpha_c T$ units of time and Σ_2 for $(1 - \alpha_c)T$ units of time, with sufficiently small time interval T leads to an unstable switched system.

(ii) \Rightarrow (iii): Assume that the matrix pencil $\alpha A_1 + (1 - \alpha)A_2$ is Hurwitz for all $\alpha \in [0, 1]$. It can be shown that $\alpha A_1 + (1 - \alpha)A_2^{-1}$ is also Hurwitz for all $\alpha \in [0, 1]$. Hence, by the result in [8], a CQLF exists for Σ_1 and Σ_2 .

(iii) \Rightarrow (i): Trivial. \square

B. Proof of Lemma 1

The assertions (i) \Rightarrow (ii) and (iii) \Rightarrow (i) follow from Proposition 1. In order to prove (ii) \Rightarrow (iii), suppose that the matrix pencil achieves its largest real eigenvalue $\lambda < 0$ for $\alpha = \alpha_c \in [0, 1]$ and $A = \alpha_c A_1 + (1 - \alpha_c)A_2$. Let $Av = \lambda v$ and $A^T w = \lambda w$. Since $-A$ is an M -matrix, it can be shown that $v, w \succ 0$ and $w = [v_2, v_1]^T$ where $v = [v_1, v_2]^T$ [1]. Define a diagonal matrix $D = \text{diag}[w_1/v_1, w_2/v_2]$. From $A^T w = \lambda w \prec 0$ and $Av = \lambda v \prec 0$, it follows that $(A^T D + DA)v \prec 0$. Hence, $-(A^T D + DA)$ is an M -matrix [1] and since it is also symmetric, $(A^T D + DA) < 0$, and D is a Lyapunov solution for A . We now proceed to show that it is also a Lyapunov solution for A_1 and A_2 . We consider two cases: (i) $\alpha_c \in \{0, 1\}$ and (ii) $\alpha_c \in (0, 1)$.

Case (i): Without loss of generality, assume $\alpha_c = 1$.

Denote the matrices as $A_i = \begin{bmatrix} -1 & a_i \\ b_i & -1 \end{bmatrix}$, $i = 1, 2$. The largest eigenvalue of A_i is $\lambda_i = -1 + \sqrt{a_i b_i}$. As $\alpha_c = 1$, $\lambda_1 \geq \lambda_2$, hence $a_1 b_1 \geq a_2 b_2$. This may happen when (a) $A_1 \succeq A_2$ (i.e. $a_1 \geq a_2, b_1 \geq b_2$), (b) Otherwise, i.e. when either $(a_1 < a_2, b_1 > b_2)$ or $(a_1 > a_2, b_1 < b_2)$.

The first subcase (a), ($A_1 \succeq A_2$) is trivial; as any diagonal Lyapunov matrix for A_1 would be a Lyapunov matrix for A_2 . Consider the 2nd subcase (b). Here, $(a_1 - a_2)(b_1 - b_2) < 0$. The largest eigenvalue of the matrix pencil, when given as a function of α , achieves its maximum for $\alpha_{max} = p/q$ where $p = a_1 b_2 + a_2 b_1 - 2a_2 b_2$ and $q = -2(a_1 - a_2)(b_1 - b_2)$. As $\alpha \in [0, 1]$, so $\alpha_c = 1$ when $\alpha_{max} \geq 1$. Hence, $\alpha_c = 1$, if $p \geq q > 0$, i.e., we have

$$a_1 b_2 + a_2 b_1 \leq 2a_1 b_1 \quad \text{and} \quad 0 \leq a_2 b_2 \leq a_1 b_1 < 1. \quad (3)$$

The eigenvectors of A_1 and A_1^T corresponding to λ_1 can be computed as $v = [\frac{1}{\sqrt{b_1}}, \frac{1}{\sqrt{a_1}}]^T$ and $w = [\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{b_1}}]^T$. Define $D = \text{diag}[w_1/v_1, w_2/v_2]$. As before, D is a Lyapunov matrix for A_1 . Using (3), $(A_2^T D + DA_2)v \prec 0$ and hence, D is a Lyapunov matrix for A_2 as well.

Case (ii): $\alpha_c \in (0, 1)$. Let $A = \alpha_c A_1 + (1 - \alpha_c)A_2$, and consider the matrix pencil $B = \beta A + (1 - \beta)A_1$, $\beta \in [0, 1]$ which achieves its largest real eigenvalue for $\beta = 1$. Hence, this case reduces to case (i) discussed above, and a diagonal common Lyapunov matrix, D , can be calculated for A and A_1 using the eigenvectors of A . Similar arguments hold for the matrices, A and A_2 . Thus, D is a Lyapunov matrix for both A_1 and A_2 . \square

C. Proof of Theorem 1

The assertions (i) \Rightarrow (ii), (ii) \Rightarrow (iii) and (v) \Rightarrow (i) are straightforward. The claim (iii) \Rightarrow (iv) follows from the equivalence of the items (ii) and (iii) in Lemma 1. In order to prove (iv) \Rightarrow (v), we proceed as follows. Let every pair of subsystems Σ_i and Σ_j , $i, j = 1, 2, \dots, N$, $i \neq j$, have a diagonal CQLF. The subsystem Σ_i has a set of normalized diagonal Lyapunov matrices $D_i = \text{diag}[1, d_i]$, where d_i can be seen to lie in a convex set, $d_i \in (d_i^-, d_i^+)$, $d_i^- \geq 0$, $d_i^+ > 0$ [8], [4]. Similarly for Σ_j , we have $D_j = \text{diag}[1, d_j]$, $d_j \in (d_j^-, d_j^+)$, $d_j^- \geq 0$, $d_j^+ > 0$. As Σ_i and Σ_j have a diagonal CQLF, the set $(d_i^-, d_i^+) \cap (d_j^-, d_j^+)$ is non-empty. Since this is true for every pair of subsystems, we have $\bigcap_{i=1}^N (d_i^-, d_i^+) \neq \emptyset$ which establishes that there exists a diagonal CQLF, $x^T D x$, for all of the subsystems. Note that the set of common Lyapunov functions can be computed by intersecting the intervals (d_i^-, d_i^+) whose limits can be determined by solving a second order algebraic equation for each matrix A_i . An alternative solution is Algorithm 1 which follows from the proof of Lemma 1. (QED) \square

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