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Stability of unfalsified adaptive control with non-SCLI controllers and related performance under different prior knowledge

Claudia Manuelli, Seung Gyun Cheong, Edoardo Mosca and Michael G. Safonov

Abstract— **The paper addresses two aspects related to adaptive switching supervisory control. The first pertains to the study of cost-detectable cost functions whose minimization guarantees internal adaptive stability of the controlled system, whenever stabilization is feasible. In this context, a new class of cost functions which removes the SCLI controller requirement in order to have cost-detectability is presented. The second aspect is the analysis of switching performance achieved using such cost functions. Both aspects are considered in the case where no a priori knowledge on the plant is available as well when nominal approximating models of the plant are known.**

I. Introduction

Adaptive control is usually employed to control time varying and/or unknown plants. In particular, the level of knowledge of the plant available to the designer varies among different applications. A general adaptive control system is depicted in Fig. 1 where P is the plant to be controlled, $\hat{C}$ is the adaptive one degree of freedom controller whose parameters are chosen by the adjustment mechanism S based on the plant input u and output y. The notation $(P/\hat{C})$ will be used to refer to the feedback control system composed by the plant P and the controller $\hat{C}$.

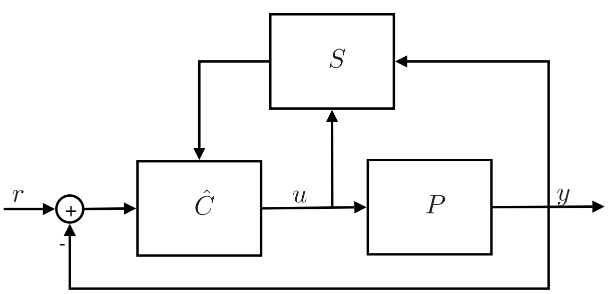

Fig. 1. Basic scheme of an adaptive control system.

This paper is focused on issues arising from the use of a *switching supervisory control* to orchestrate the general adaptive controller in the scheme of Fig. 1. The adaptive controller $\hat{C}$ can only take on values from the candidate controller set $\mathcal{C} = \{C_1, \ldots, C_N\}$, assumed to have a finite number of elements. If the switching stops, then the final controller and the final switching time are respectively indicated by C^f and t^f. The block S, called *supervisor*, is responsible for the switching process: it assesses the performance of both the current controller as well the ones of the "idle" candidate controllers and it decides when and to which controller to switch. The two main tasks of the supervisor arising from its operative lines are: *controller falsification* [1], [2], [4] and *inference of candidate loop behavior* [3]. Controller falsification deals with the problem of deciding whether the current controller is adequate or not for the actual operating conditions. Inference of candidate loop behavior addresses the problem of inferring the behavior of a potential loop made up by a candidate controller connected in feedback to the plant. Loop behavior of candidate loops should be inferred without directly checking the behavior of each candidate controller via its effective use in the control system. The supervisor commonly achieves these goals through the minimization of a cost function $V : \mathcal{C}\text{x}\mathbb{D}\text{x}\mathfrak{R} \rightarrow \mathfrak{R}$ where $\mathbb{D}$ is the set of all possible experimental plant data $d = [u \quad y]'$. Throughout the paper, the supervisor is assumed to be based on an ε hysteresis switching logic of the form:

$$\sigma(t) = \arg\min_i \{V_i(t) - \varepsilon\delta_{i\sigma(t-1)}\} \tag{1}$$

where $\varepsilon > 0$, $\sigma(t) = \{1, \ldots, N\}$ is the index of the next candidate controller to be put in the loop, δ_{ij} is the Kronecker's δ, and V is the cost function chosen for the problem under study.
According to the supervisory tasks, the cost function should ideally reflect the ability of a controller to yield good performance when switched on in feedback in the loop. Different cost functions can be designed according to the prior knowledge available to the designer. In particular Sec. II pertains to the case where no a priori knowledge on the plant is available, while Sec. III analyzes cost functions which exploit the availability of nominal approximating models.

II. Unfalsified adaptive control

The problem of controlling a completely unknown plant is dealt in this section within a framework commonly referred to as *unfalsified control* [2]. Although useful definitions and results are available in literature [5], [6], some of them are reported here for the reader's convenience.
Let $\mathfrak{R}_+ = (0, \infty)$, all the involved signals are commonly assumed to belong to $\mathfrak{L}_{2e}$, viz. to be square integrable over any bounded interval of time $[0, t]$, $t \in \mathfrak{R}_+$.

Definition 2.1: Given plant data $d = [u \quad y]'$ and a candidate controller C, a *fictitious or virtual reference signal* [2], [3] $\bar{r}_C$ is the reference signal that would have produced data $d = [u \quad y]'$ had the candidate controller C been in feedback

C. Manuelli and E. Mosca are with Dipartimento di Sistemi e Informatica - Università di Firenze V. di S. Marta 3, 50139 Firenze, Italy {manuelli, mosca}@dsi.unifi.it
S.G. Cheong and M.G. Safonov are with the Dept. of Electrical Engineering, University of Southern California, Los Angeles, CA msafonov@usc.edu

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to the plant during the entire time period over which d were collected.
From this definition the following expression for the virtual reference signal follows:

$$\bar{r}_C(t) = y(t) + C^{-1}u(t) \tag{2}$$

Definition 2.2: A controller C with input (r, y) and output u is said *Stably Causally Left Invertible* (SCLI) if the map $d \mapsto \bar{r}_C$ exists, is causal and incrementally stable.
A necessary condition for the controller C to be SCLI is to be minimum-phase and biproper. Key definitions in this approach are the following:

Definition 2.3: A system G with input v and output w is said to be *stable* if for every input $v \in \mathfrak{L}_{2\mathfrak{e}}$ there exist constants β, $\alpha \geq 0$ such that:

$$\|w^t\| < \beta \|v^t\| + \alpha, \forall t > 0 \tag{3}$$

where $\|x^t\|^2 = \int_0^t |x(\tau)|^2 d\tau$ and x^t is the time truncation of $x(t)$ over the interval $(0, t)$ and $|x|$ is the Euclidean norm. Otherwise, it is said to be *unstable*. Furthermore, if (3) holds with a single pair $\beta, \alpha \geq 0$ for all $v \in \mathfrak{L}_{2\mathfrak{e}}$, then the system G is said to be *finite-gain stable*, in which case the *gain* of G is the least such β.

Definition 2.4: Given an input-output pair (v, w) of a system, the stability of the system is said *unfalsified* by (v, w) if there exist bounded $\beta, \alpha \geq 0$ such that (3) holds. Otherwise, stability of the system is *falsified* by (v, w).

Definition 2.5: Let r denote the input and let $d = [u \quad y]'$ denote the resulting plant data collected while $\hat{C}$ is in the loop. The pair $(V, \mathcal{C})$ is said to be *cost-detectable* if for every $\hat{C} \in \mathcal{C}$ with finitely many switching times, the following statements are equivalent: 1) $V(C^f, d, t)$ is bounded as $t \to \infty$, 2) stability of the system $(P/\hat{C})$ is unfalsified by (r, d).

Definition 2.6: Given the pair $(V, \mathcal{C})$, V is said to be $\mathfrak{L}_{2\mathfrak{e}}$-*gain-related* if for each $d \in \mathfrak{L}_{2\mathfrak{e}}$ and $C \in \mathcal{C}$: 1) $V(C, d, t)$ is monotone in t, 2) the virtual reference signal $\bar{r}_C$ exists, and 3) $\forall C \in \mathcal{C}$ and $d \in \mathfrak{L}_{2\mathfrak{e}}$, $V(C, d, t)$ is bounded as $t \to \infty$ if and only if stability is unfalsified by $(\bar{r}_C, d)$.

A. SCLI controllers

Cost-detectability enables cost functions to reliably detect any instability exhibited by the adaptive system. An example of cost-detectable cost function is the following:

$$V_C(t) = \max_{\tau \leq t} \frac{\| e_C^\tau \|^2 + \rho \| u^\tau \|^2}{\| \bar{r}_C^\tau \|^2} \tag{4}$$

where $\tau = 0$ is the instant when the loop has been closed. The signal $e_C(t) = \bar{r}_C(t) - y(t) = C^{-1}u(t)$ is the virtual tracking error, i.e., the tracking error that would have occurred had the controller C been in the loop and the reference equal to $\bar{r}_C$. It can be easily proved that (4) is $\mathfrak{L}_{2\mathfrak{e}}$-gain-related [5], hence, assuming SCLI controllers, cost-detectability of (4) is guaranteed by Th. 1 in [5]. Moreover, use of the virtual reference allows calculation of the cost function related to each candidate controller without physically inserting it in feedback to the plant. Notice that only input-output data from the currently operating loop are required for cost function calculation.

B. Non-SCLI controllers

In the case of nonminimum phase controllers the map $d \mapsto \bar{r}_C$ is unstable and it involves numerical problems as well as loss of cost-detectability of the cost function in (4). In order to recover this property a different cost function has been studied and the following definition is needed:

Definition 2.7: The pair (N_C, D_C) is a *left Matrix-Fraction Description (MFD)* of the controller C if N_C and D_C are stable, D_C is invertible and $C = D_C^{-1} N_C$.
Suppose to virtually put the factor $D_{C_0}^{-1}$ of the operating controller C_0 in the forward path and the factor N_{C_0} in the backward path of the closed loop, then the scheme in Fig. 2 is obtained. Following the same conceptual lines of Def. 2.1, it

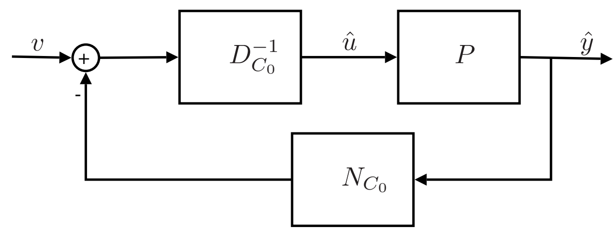

Fig. 2. Operating feedback loop when the controller is written in the MFD form

is possible, given plant data $\hat{d} := [\hat{u} \quad \hat{y}]'$ and a controller C, to define a "new" virtual reference signal $\bar{v}_C(t)$, according to the virtual configuration shown in Fig. 3:

$$\bar{v}_C(t) = N_C \hat{y}(t) + D_C \hat{u}(t) \tag{5}$$

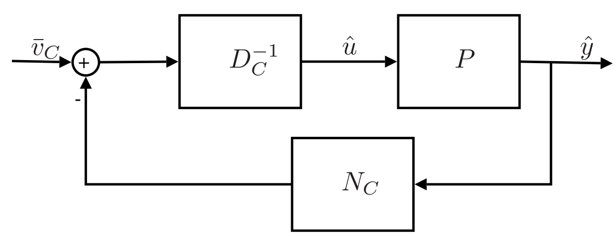

Fig. 3. Virtual closed loop associated to the definition of $\bar{v}_C(t)$.

It is worth pointing out that the use of stable matrix fraction descriptions to deal with nonminimum phase controllers is not new in literature, see [7], [8]. In this context, the contribution of the present work is to prove that SCLI adaptive controllers are internally stable in the presence of cost-detectable cost functions. By this way, no reconfiguration of the control system in order to actually have the factor $N_{\hat{C}}$ of the adaptive controller in the feedback path is needed.
To this end, a cost function of the form:

$$\hat{V}_C(t) = \max_{\tau \leq t} \frac{\|w_C^\tau\|^2 + \rho \|\hat{u}^\tau\|^2}{\|\bar{v}_C^\tau\|^2} \tag{6}$$

where $w_C(t) = \bar{v}_C(t) - \hat{y}(t)$ and $\bar{v}_C(t)$ is given by (5) recovers the cost-detectability property. Such property is ensured by Th. 1 in [5] because *fictitious* controllers $\hat{u}(t) =$

$D_C^{-1}(v(t) - N_C\hat{y}(t))$ are SCLI in accordance with Def. 2.2 and hence, $\mathfrak{L}_{2e}$-gain-relatedness of $\hat{V}_C$ can be easily verified $\forall \hat{d} \in \mathfrak{L}_{2e}$ and $\forall C \in \mathcal{C}$. Controllers in the form of Fig. 2 are referred to as *fictitious* because, as a consequence of next theorem, they are not actually implemented in practice.

Theorem 1: Suppose that the control problem is feasible. Then, the adaptive closed loop system in Fig. 4 is internally stable.

Proof: By Th. 1 in [5] cost-detectability of (6) holds. Further, by Th. 2 in [5] cost-detectability of (6) and problem feasibility ensure that the switching process stops in finite time and stability of the adaptive closed loop system of Fig. 4, where the supervisor block has been omitted, is not falsified by any possible $(v, \hat{d})$. Hence, with the adaptive

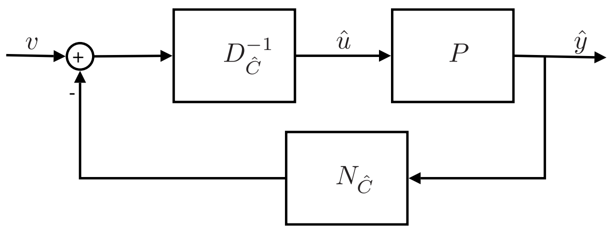

Fig. 4. Adaptive closed loop with the controller written in the MFD form.

switching loop closed, the map $T_{dv} : v \mapsto d$ is stable in accordance with Def. 2.3.

It is useful to recall that internal stability is guaranteed by stability of the map $T : [s \quad r]' \mapsto d$, where the involved signals are shown in Fig. 5. The map T is such that:

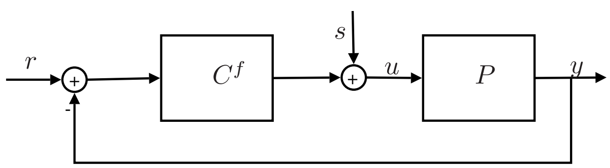

Fig. 5. Figure shows the conventional signals involved in the internal stability proof of a closed loop scheme.

$$T = T_{dv} * [D_{C^f} N_{C^f}] \tag{7}$$

and $d = T_{dv} * (D_{C^f}s + N_{C^f}r)$. Stability of the map T directly follows from stability of map T_{dv} and stability of the controller MFD factors. ∎

Internal stability of the fictitious adaptive control system of Fig. 4, stated by Th. 1, implies internal stability of the original adaptive system in Fig. 1. It allows one to implement the scheme in Fig. 1, with the adaptive controller in the forward path, even though the controllers are non-SCLI. Nevertheless, the switching process should be based on the minimization of a cost-detectable cost function of the form (6) in order to obtain internal stability of the adaptive scheme. It is worth pointing out that cost functions in (6) define a new class of cost functions which removes the controllers SCLI assumption in order to have cost-detectability.

The "new" virtual reference in (5) can be also viewed as a suitable filtered version of the conventional virtual reference in (2). The filter, F_C, filtering the virtual reference $\bar{r}_C$, should be chosen such that it is stable and cancel the nonminimum phase part of $C = D_C^{-1}N_C$. Hence, $F_C = A_C^{-1}N_C$, where A_C is stable, satisfies the requirements, giving rise to the following filtered virtual reference, called $\bar{r}_{C_F}$:

$$\begin{aligned}\bar{r}_{C_F}(t) &= F_C\bar{r}_C(t) = A_C^{-1}N_C(y(t) + (D_C^{-1}N_C)^{-1}u(t)) = \\ &= A_C^{-1}N_C y(t) + A_C^{-1}D_C u(t)\end{aligned} \tag{8}$$

Notice that expression in (5) becomes (8) once the filter is chosen to have an unit denominator.

The present approach does not make any assumption on plant linearity. Also the candidate controllers can be nonlinear too provided that they can be factored in term of nonlinear *incrementally* stable factors and we replace the linear equations (5) and (7) with their respective nonlinear generalizations (cf. [16]), viz.

$$\bar{v}_C(t) = -N_C(-\hat{y}(t)) + D_C\hat{u}(t) \tag{5'}$$

$$T = T_{dv} * \left[\ \tilde{D}_{C^f\,(u-s)} \quad \tilde{N}_{C^f\,(-y)}\ \right] \tag{7'}$$

where $\tilde{F}_{(x)}$ denotes the *incremental operator* of F at x defined for any x and $x + \Delta x$ in the domain of F by

$$\tilde{F}_{(x)}\,\Delta x = F(x + \Delta x) - F(\Delta x). \tag{9}$$

C. Analysis of the switching performance in steady-state

The purpose of this subsection is to study the performance achievable by a supervisor based on a cost function of the form (4), in the case of SCLI controllers. To this end, it should be noted that falsification of an actively operating controller that is destabilizing is always guaranteed, provided that the initial state and/or disturbances are such that unstable modes are actually excited. This is a direct consequence of the $\mathfrak{L}_{2e}$-gain-relatedness of the cost function. On the other hand, unstable data collected with a destabilizing candidate controller in the loop generally will not falsify stability of any other destabilizing candidate controller that is not actively operating in the loop. This is because for a C is not operating in the feedback loop, unstable data may cause the denominator of the cost to diverge at the same exponential rate as the numerator, so that the cost for an inactive controller will remain bounded even though that controller would be destabilizing if it were active. Thus, the ability of an inactive controller to stabilize the plant may not be falsified until after such future time as that controller is actually switched into the feedback loop. Of course as noted by Stefanovic [9], continuously parameterized controllers in the neighborhood of a destabilizing active controller will have a cost that grows very large albeit not unboundedly, whenever the cost of the active controller grows unboundedly.

Remark 2.8: Unfalsified control does not prevent non-stabilizing candidate controllers from being switched on in the loop. Nevertheless, it guarantees that, once in operation, they are removed as soon as the instability is detected.

In what concerns the inference of candidate loop behavior capability, a steady-state analysis of the cost function, in the absence of noise, is a prerequisite for further studies addressing more realistic operating conditions. To this aim, suppose that all involved signals be stationary, and $C_0 \in \mathcal{C}$ be the current stabilizing controller switched on in the loop since the remote past (see Fig. 6). Then the following results can be easily obtained:

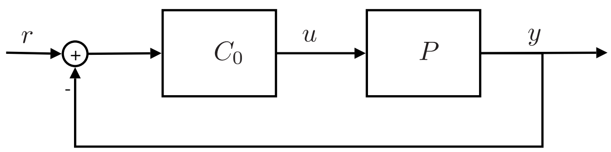

Fig. 6. Current loop

$$\begin{aligned} \bar{r}_C(t) &= \frac{C_0}{C}\frac{1+PC}{1+PC_0}r(t) = \frac{\mathcal{T}_0}{\mathcal{T}}r(t) \\ e_C(t) &= -\frac{\mathcal{T}_0}{\mathcal{T}}(\mathcal{T}-1)r(t) \\ V_C|_{\rho=0} &= \lim_{t\to\infty}\frac{\| e_C^t \|^2}{\| \bar{r}_C^t \|^2} = \frac{\| \frac{\mathcal{T}_0}{\mathcal{T}}(\mathcal{T}-1)r^t \|^2}{\| \frac{\mathcal{T}_0}{\mathcal{T}}r^t \|^2} = \\ &= \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}|\frac{\mathcal{T}_0}{\mathcal{T}}(\mathcal{T}-1)|^2\Phi_r(\omega)d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty}|\frac{\mathcal{T}_0}{\mathcal{T}}|^2\Phi_r(\omega)d\omega} \end{aligned} \tag{10}$$

where the argument $j\omega$ is omitted in all transfer functions, $\Phi_r(\omega)$ is the power spectral density of the reference signal $r(t)$, $\mathcal{T}_0$ and $\mathcal{T}$ are the complementary sensitivity functions of the (constant) operating loop (P/C_0) and respectively of the generic candidate loop (P/C). For the sake of simplicity the analysis is carried out for $\rho = 0$. Nevertheless, similar conclusions can be obtained for $\rho > 0$. Eq. (10) involves some positive inference feature as it depends on the square of the difference between $\mathcal{T}$ and 1, its desired value, weighted by the normalized dynamic weight $|\mathcal{T}_0/\mathcal{T}|^2\Phi_r(\omega)$. However, it should be observed that, being the weight smaller at those frequencies where $|\mathcal{T}| \gg 1$, unsuitable candidate controllers may have a greater chance to be selected. Simulations have been carried out and confirm the qualitative indication of the former analysis.

III. Multiple Approximating models

Previous section shows that the adaptive supervisor does not require any plant model to choose an unfalsified-cost-minimizing controller from the candidate controller set. However, plant models actually play a key role in determining what the candidate controller set should be. Moreover, the supervisor can accomplish its tasks taking advantage from the plant models knowledge . These are some of the reasons for the present section to address situations where some a priori knowledge on the unknown plant can be exploited by the designer. In particular, the present discussion considers a set of approximating models of the plant P, viz. $\mathcal{M} = \{M_{C_1}, \dots, M_{C_N}\}$. $\mathcal{M}$ can be viewed as a set of open loop models, for instance representing different operating conditions. Each candidate controller $C_i \in \mathcal{C}$ is supposed to be tuned on the corresponding model M_{C_i} in such a way that the reference loop (M_{C_i}/C_i) is obtained.
Even though the use of multiple models to identify and then to control the plant is by no means new (see e.g. [10], [11]), previous [12] and more recent works [13], show that the model-plant mismatch is a potential problem of this approach as it may cause destabilization effects.
In this context, the contribution of the present work is to design a cost-detectable cost function which, taking only input-output data from the closed loop system, ensure controller falsification and a good inference of candidate loops behavior.

A. SCLI controllers

In [4] controller falsification is dealt with a statistical ratio to be compared with a pre computed threshold in order to detect instability or poor performance trend. A different falsification approach is presented in this context.
In order to quantitatively evaluate the behavior of a generic candidate loop (P/C), the scheme in Fig. 7 is useful [1]. In

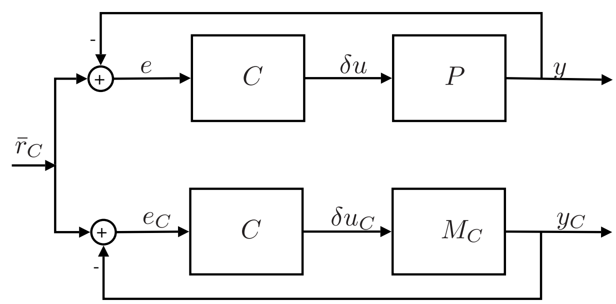

Fig. 7. Double closed loop related to controller C with the virtual reference signal $\bar{r}_C$ as the common reference signal.

fact it shows the same reference signal driving two closed loops: the candidate loop (P/C) and the corresponding nominal loop (M_C/C). As the two loops differ only for the plant block, any difference exhibited by their outputs (y, y_C) or their inputs (δu, δu_C, being δu the input derivative) has to be caused by the discrepancy between P and M_C. The reason for using input derivatives is mainly to enforce an integral action by the controller.
Being intuitive that the closer (P/C) behaves to (M_C/C), the more adequate is the use of C for controlling P, a cost function which reflect the discrepancy between the two loops appears suitable to be minimized by the supervisory logic.
Hence, the following cost function is proposed [14]:

$$J_C(t) = \max_{\tau\le t}\frac{\| \eta_{C_y}^\tau \|^2 + \rho \| \eta_{C_u}^\tau \|^2}{\| \bar{r}_C^\tau \|^2} \tag{11}$$

where $\eta_{C_y}(t) = y(t) - y_C(t)$, $\eta_{C_u}(t) = \delta u(t) - \delta u_C(t)$ and $\bar{r}_C(t)$ is the virtual reference obtained by (2). The presence of the virtual reference in the cost function denominator has several advantages. First, it normalizes the numerator in such a way that the whole cost function does not depend on the level of $\bar{r}_C$ which may change among different controllers. Notice that the true reference $r(t)$ could have driven the double loop of Fig. 7 in order to obtain a cost function still suitable for measuring the discrepancy between closed loops, without any normalization be necessary. Nevertheless, it would have required the insertion of each candidate controller in the loop (pre-routing) in order to calculate $[\delta u \quad y]'$, involving an unsafe way of inferring the controllers behavior. On the contrary, Def. 2.1 allows one to take data $[\delta u \quad y]'$ from the operating loop avoiding the undesirable real implementation of the upper loop in Fig. 7. The scheme in Fig. 8 shows the supervisory logic necessary to produce all the signals involved in (11). A parallel implementation of

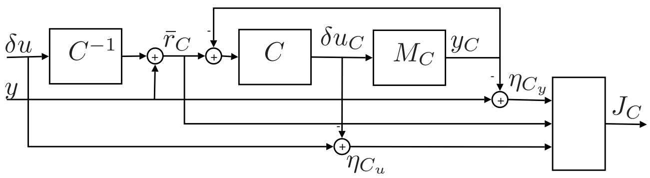

Fig. 8. Figure shows the supervisory logic necessary to produce the cost function J_C associated to the controller C.

the same logic for the N candidate controllers permits the switching logic to determine the next controller to be put in the loop, according to (1), only providing input-output data $[\delta u \quad y]'$ from the operating loop.
A different cost function is obtained choosing a different normalization:

$$\tilde{J}_C(t) = \max_{\tau \leq t} \frac{\| \eta^\tau_{C_y} \|^2 + \rho \| \eta^\tau_{C_u} \|^2}{\| e^\tau_C \|^2 + \rho \| \delta u^\tau_C \|^2} \qquad (12)$$

where $e_C(t) = \bar{r}_C(t) - y_C(t)$, $y_C(t)$ and $\delta u_C(t)$ are the input and the output of the nominal loop (M_C/C) driven by $\bar{r}_C$. Roughly speaking, the cost function in (12) measures a relative discrepancy between the behaviors of the two loops in Fig. 7. In fact, the denominator can be viewed as a measure of the nominal loop behavior. All the signals in (12) can still be obtained from the scheme in Fig. 8, and all the conlusions already stated for the cost function in (11) are valid for that in (12).
Cost-detectability of the pair $(J, \mathcal{C})$ is guaranteed by Th. 1 in [5], once SCLI controllers are provided and $\mathfrak{L}_{2\mathfrak{e}}$-gain-relatedness of the cost function in (11) is proved.

Proposition 3.1: The cost function $J_C(t)$ in (11) is $\mathfrak{L}_{2\mathfrak{e}}$-gain-related.

Proof: The controllers are supposed to be SCLI, hence the virtual reference generator exists. Moreover, the presence of the max operator forces the cost function to be monotone in time, hence, according to Def. 2.6, only point 3) is left to prove $\mathfrak{L}_{2\mathfrak{e}}$-gain-relatedness. For the sake of simplicity, the analysis is restricted to a linear SISO model of the form $M_C = B_C/A_C$ and a linear SISO controller $C = S_C/R_C$. Let, $\lim_{t\to\infty} J_C(t)$ be bounded by hypothesis, then, being $J_C(t)$ monotone, a constant $M < \infty$ exists, such that $J_C(t) < M, \quad \forall t \geq 0$. It is equivalent to the existence of $M_1, M_2 < \infty$ such that:

$$\frac{\|\eta^t_{C_y}\|^2}{\|\bar{r}^t_C\|^2} < M_1 \quad , \quad \frac{\|\eta^t_{C_u}\|^2}{\|\bar{r}^t_C\|^2} < M_2$$

Hence, the triangular inequality applied to the first part yields:

$$\|y^t - y^t_C\| < M_1^{1/2}\|\bar{r}^t_C\| \Rightarrow \|y^t\| < M_1^{1/2}\|\bar{r}^t_C\| + \|y^t_C\| \Rightarrow$$
$$\|y^t\| < M_1^{1/2}\|\bar{r}^t_C\| + \left\|\frac{B_C S_C}{\chi_C}\bar{r}^t_C\right\|$$

where $\chi_C := A_C R_C + B_C S_C$. Being $B_C S_C/\chi_C$ a stable transfer function by construction, then a positive real k_1 exists such that:

$$\|y^t\| < M_1^{1/2}\|\bar{r}^t_C\| + \max_\omega \left|\frac{B_C(j\omega)S_C(j\omega)}{\chi_C(j\omega)}\right| \|\bar{r}^t_C\| \Leftrightarrow$$
$$\|y^t\| < k_1\|\bar{r}^t_C\|$$

Along the same conceptual lines, it can be obtained: $\|\delta u^t\| < k_2\|\bar{r}^t_C\|$. It proves that stability is not falsified by $(\bar{r}_C, [\delta u \quad y]')$.
Suppose now that stability is not falsified by $(\bar{r}_C, [\delta u \quad y]')$, viz. there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ such that, $\forall t > 0$:

$$\|y^t\| < \beta_1\|\bar{r}^t_C\| + \alpha_1 \quad , \quad \|\delta u^t\| < \beta_2\|\bar{r}^t_C\| + \alpha_2 \qquad (13)$$

Taking into account stabiliy of the closed loop (M_C/C) by construction, a positive real z_1 exists such that:

$$\|y^t_C\| \leq \max_\omega \left|\frac{B_C(j\omega)S_C(j\omega)}{\chi_C(j\omega)}\right| \|\bar{r}^t_C\| \Leftrightarrow \|y^t_C\| \leq z_1\|\bar{r}^t_C\|$$

Combining this last equation with the first inequality in (13) and provided that $\bar{\beta}_1 = \beta_1 + z_1$, it results $\forall t > 0$:

$$\|y^t\| + \|y^t_C\| \leq \bar{\beta}_1\|\bar{r}^t_C\| + \alpha_1$$

Using then the triangular inequality, it follows:

$$\|y^t - y^t_C\| \leq \bar{\beta}_1\|\bar{r}^t_C\| + \alpha_1 \Leftrightarrow$$
$$\frac{\|y^t - y^t_C\|}{\|\bar{r}^t_C\|} \leq \bar{\beta}_1 + \frac{\alpha_1}{\|\bar{r}^t_C\|} =: \gamma_1(t) \Leftrightarrow \frac{\|\eta^t_{C_y}\|^2}{\|\bar{r}^t_C\|^2} \leq (\gamma_1(t))^2$$

Along the same conceptual lines, it can be proved that:

$$\frac{\|\eta^t_{C_u}\|^2}{\|\bar{r}^t_C\|^2} \leq (\gamma_2(t))^2 \quad \forall t > 0$$

The last two inequalities directly lead to boundedness of $\lim_{t\to\infty} J_C(t)$. ■
Hence, under the assumption of problem feasibility, Th. 2 in [5] guarantees internal stability of the adaptive system in Fig. 1 once the switching process is based on (11).
On the contrary, $\mathfrak{L}_{2\mathfrak{e}}$-gain-relatedness of cost function in (12) can not be proved. In order to recover that property, a variant of (12) consisting of an approximation of the denominator in (12) is considered [3]. To this end, the following results can be easily obtained:

$$\begin{aligned} e_C(t) &= \frac{1}{1 + M_C C}\bar{r}_C(t) \\ \delta u_C(t) &= \frac{C}{1 + M_C C}\bar{r}_C(t) \end{aligned} \qquad (14)$$

Moreover, supposed that $\bar{r}_C(t)$ be a stationary process with power spectral density $\Phi_{\bar{r}_C}(\omega)$, then, omitting the argument $j\omega$ for all transfer functions, it results:

$$\lim_{t\to\infty}(\|e^t_C\|^2 + \rho\|\delta u^t_C\|^2) =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1 + \rho|C|^2}{|1 + M_C C|^2}\Phi_{\bar{r}_C}(\omega)d\omega$$

Let $W_C = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+\rho|C|^2}{|1+M_CC|^2}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}|T_C|^2 d\omega$. W_C can be computed off line in that it is unaffected by experimental data, only depending on the nominal loop composed by the model M_C and the controller C. Notice that $|T_C|$ is the norm of the mixed sensitivity of the loop (M_C/C).

Previous reasoning leads to the following approximation of the cost function in (12):

$$\bar{J}_C(t) = \max_{\tau\leq t}\frac{\| \eta^{\tau}_{C_y} \|^2 + \rho \| \eta^{\tau}_{C_u} \|^2}{W_C \| \bar{r}^{\tau}_C \|^2} \tag{15}$$

Notice that (15) recovers the $\mathfrak{L}_{2\mathfrak{e}}$-gain-relatedness property and gives encouraging insights on a good performance of the switching criterion, as it will be shown shortly.

B. Analysis of the switching performance in steady-state

Falsification of a currently operating controller leading to divergent data, is a consequence of Prop. 3.1. On the contrary, falsification of candidate controllers virtually yielding unstable loops, is not guaranteed as long as they are out of the loop (see first paragraph of Sec. II-C). Moreover, cost functions in (11) or (15) do not prevent such candidate controllers from being switched on in the loop as it has been already highlighted in Remark 2.8. However, the steady-state analysis that follows shows that there are good chances to obtain a satisfactory performance. Suppose that all the involved signals are stationary, a steady-state analysis of the cost function in (15), in the absence of noise, is carried out. To this aim, the following results can be easily obtained [14]:

$$\begin{aligned} \eta_{C_y}(t) &= \frac{\tilde{P}_C C}{1+PC}\frac{1}{1+M_CC}\bar{r}_C(t) \\ \eta_{C_u}(t) &= -\frac{\tilde{P}_C C}{1+PC}\frac{C}{1+M_CC}\bar{r}_C(t) \end{aligned} \tag{16}$$

where $\tilde{P}_C = P - M_C$. Consequently,

$$\begin{aligned} \bar{J}_C &= \lim_{t\to\infty}\frac{\| \eta^{t}_{C_y} \|^2 + \rho \| \eta^{t}_{C_u} \|^2}{W_C \| \bar{r}^{t}_C \|^2} = \\ &= \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{P}_C C}{1+PC}\right|^2 \frac{1+\rho|C|^2}{|1+M_CC|^2}\Phi_{\bar{r}_C}(\omega)d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+\rho|C|^2}{|1+M_CC|^2}d\omega\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{\bar{r}_C}(\omega)d\omega} = \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{\tilde{P}_C C}{1+PC}\right|^2 \overline{|T_C|^2}\quad \overline{\Phi}_{\bar{r}_C}(\omega)d\omega \end{aligned} \tag{17}$$

where:

$$\begin{aligned} \overline{|T_C|^2} &= \frac{\frac{1+\rho|C|^2}{|1+M_CC|^2}}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1+\rho|C|^2}{|1+M_CC|^2}d\omega} \\ \overline{\Phi}_{\bar{r}_C} &= \frac{\Phi_{\bar{r}_C}}{\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi_{\bar{r}_C}(\omega)d\omega} \end{aligned}$$

and $\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{|T_C|^2}d\omega = 1$, $\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\Phi}_{\bar{r}_C}d\omega = 1$. Hence, the quantity $\left|\frac{\tilde{P}_C C}{1+PC}\right|^2$ in (17) is weighted by the "normalized" dynamic weight $\overline{|T_C|^2}\quad\overline{\Phi}_{\bar{r}_C}(\omega)$ which has the positive feature to be greater at those frequencies where the mixed sensitivity function of the nominal loop (M_C/C) is higher. In order to further analyze (17), it is worth recalling a robust stability property (see i.e. [15]). It states that, given a stable nominal loop (M_C/C), a sufficient condition for stability of the loop (P/C) is the following:

$$\left|\frac{\tilde{P}_C(j\omega)C(j\omega)}{1+M_C(j\omega)C(j\omega)}\right| < 1 \quad \forall\omega \tag{18}$$

provided that M_CC and PC have the same number of unstable poles. Notice that: $\left|\frac{\tilde{P}_C C}{1+PC}\right| = \left|\frac{\tilde{P}_C C}{1+\tilde{P}_C C+M_CC}\right| \simeq \left|\frac{\tilde{P}_C C}{1+M_CC}\right|$ where the last approximation holds, provided that:

$$\left|\frac{\tilde{P}_C(j\omega)C(j\omega)}{1+M_C(j\omega)C(j\omega)}\right| \ll 1 \quad \forall\omega \tag{19}$$

Then, if the model set $\mathcal{M}$ is chosen in such a way that, for any given (expected) P, there is at least an M_C so as to satisfy (19), minimizing (17) may amount to minimizing the quantity on the LHS of (18). Thus, minimization of a cost function, whose steady-state value is shown in (17), does not imply the robust stability property (18), but gives encouraging insights on the related controller selection criterion.

C. Non-SCLI controllers

In the case of nonminimum phase controllers, the scheme in Fig. 8 presents some difficulty because of the inversion of a non-SCLI controller. It may lead to numerical problems in the virtual reference calculation as well as loss of cost-detectability of cost function in (15). Hence, a different way to calculate both numerator and denominator of (15) is proposed. In what concerns the numerator, a result, firstly presented in [3], is recovered based on which signals η_{C_y} and η_{C_u} can be obtained by suitably filtering the output prediction error. More specifically, using polynomial expressions and referring, for the sake of simplicity, to a SISO case, the model can be written as $M_C = B_C/A_C$ and the controller as $C = S_C/R_C$. Let $\varepsilon_C(t) := A_C y(t) - B_C\delta u(t)$ be the output prediction error based on model M_C, given data $d = [\delta u \quad y]'$. Then:

$$\begin{aligned} \eta_{C_y}(t) &= y(t) - y_C(t) = \tfrac{R_C}{\chi_C}\varepsilon_C(t) \\ \eta_{C_u}(t) &= \delta u(t) - \delta u_C(t) = -\tfrac{S_C}{\chi_C}\varepsilon_C(t) \end{aligned} \tag{20}$$

where $\chi_C := A_C R_C + B_C S_C$. Notice that (20) does not present any trouble in the case of non-SCLI controllers. In order to deal with the denominator calculation, the quantity $\|\bar{r}^t_C\|^2$ in (15) is replaced by $\|\hat{\bar{r}}^t_C\|^2$, being $\bar{r}_C(t)$ and $\hat{\bar{r}}_C(t)$ different signals with the same energy. In fact:

$$\begin{aligned} \bar{r}_C(t) &= y(t) + \frac{R_C}{S_C}\delta u(t) = \frac{1}{S_C}(S_C y(t) + R_C\delta u(t)) \\ \hat{\bar{r}}_C(t) &= \frac{1}{\Gamma_C}(S_C y(t) + R_C\delta u(t)) \end{aligned} \tag{21}$$

where Γ_C is a solution of the spectral factorization problem: $S_C^* S_C = \Gamma_C^* \Gamma_C$, provided that S_C has no zeros on the imaginary axis.
By this way, a cost function of the form:

$$\hat{J}_C(t) = \max_{\tau \leq t} \frac{\|\eta_{C_y}^{\tau}\|^2 + \rho \|\eta_{C_u}^{\tau}\|^2}{W_C \|\hat{\tilde{r}}_C^{\tau}\|^2} \tag{22}$$

where the involved signals are provided by (20) and (21) is an approximation of (15), suitable to replace (15) in the case of non-SCLI controllers. Indeed, it does not involve any numerical problem in its calculation and, moreover, it preserves all the positive properties already outlined for the SCLI controllers cost function.

IV. CONCLUSIONS

Cost-detectability is a very important property of a cost function in that it enables the supervisor to reliably detect any instability exhibited by the control system. Previous results on cost-detectability required controllers to be SCLI. This paper presents a new class of cost-detectable cost functions which do not require the SCLI assumption. In this new class, the original virtual reference signal is replaced with a modified virtual reference signal synthesized via the controller MFD stable factors.
The cost-detectability concept is by definition plant-independent in that it does not require any assumption on the plant to be verified. Nevertheless, Sec. III of the paper shows that cost-detectable cost functions can exploit the availability of a priori known approximating models of the plant.
Internal stability of the hysteresis switched adaptive control system, designed using cost-detectable cost functions, is proved.
Besides internal stability, the cost function used for adaptive switching control should ensure good performance. A steady-state analysis of the cost function performance capabilities has been carried out.

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