

Real/Complex K_m -Synthesis without Curve Fitting

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I Introduction

The K_m -synthesis theory concerns the synthesis of multivariable feedback control laws with a robust tolerance of uncertain variations in the gains and or phases in several, possibly multivariable, feedback loops. The term “ K_m ” refers to the multivariable stability margin [1, 2]; it is the reciprocal of the structured singular value μ [3], i.e., $\mu = 1/K_m$. The term “synthesis” refers to the synthesis (i.e., automatic design) of feedback control laws. Thus, K_m -synthesis and μ -synthesis concern the automatic design of control laws with good multivariable stability margins.

The concept of multivariable stability margin has a history that goes back nearly thirty years. Though not focusing on the issues of uncertainty or stability margins, the 1960’s input-output stability results of Sandberg [4] and Zames [5] based on conic-sectors, positivity and loop-gain contain the key concepts and formed the basis for most modern approaches to multivariable stability margin analysis. Without specific reference to the term “singular value,” the small-gain nonlinear stability results of [4, 5] both incorporated singular value conditions of the sort now popularly used to evaluate multivariable stability margins for linear time-invariant systems. Taking a different tack, Rosenbrock made use of diagonal dominance conditions to evaluate the stability robustness of multiloop feedback control systems against simultaneous variations in the gains in several feedback loops (see [6] and the references therein). The concept of multivariable stability margin was introduced in the modern control context in [7, 1]. The connection between the nonlinear-stability results of the Sandberg/Zames type involving conic-sectors and singular values was made by Safonov and

Athans [7, 8, 9]. The singular value approach has been further developed by a number of authors including, for example, [10, 2, 11, 12, 13].

A quantitative measure of multivariable stability margin, called the *excess stability margin* K_m , was introduced by Safonov and Athans [2]. Safonov [14, 15] developed a technique based on Perron eigenvalues/eigenvectors to optimize diagonal scalings so as to produce less conservative estimates of K_m than would be possible with singular values alone. The terminology *structured singular value* μ was introduced by Doyle [3]; μ is the reciprocal of Safonov and Athans' multivariable stability margin K_m . Doyle [3] showed that optimal diagonally scaled singular values produce a nonconservative estimate of μ for systems with three complex uncertainty blocks. Algorithms for optimal diagonal scaling and generalizations thereof were further studied by [16, 17]. Techniques for further reducing conservativeness of μ computations for systems with one or more uncertain real gains were introduced by Doyle [18] and further developed by Fan et al [19]. Safonov and Lee [20] developed a multiplier formulation of these results and associated computational algorithms based on the preliminary work of Chiang and Safonov [21].

The concept of K_m -synthesis, or μ -synthesis, was introduced in the papers of Safonov [22] and Doyle [23]. A hybrid of the H^∞ control theory (e.g., [24, 25]) and the diagonal scaling techniques for multivariable stability margin analysis, K_m -synthesis is potentially broadly applicable theory for the synthesis of multiloop feedback control systems that perform robustly despite H^∞ -norm bounded uncertain gains in one or more feedback loops. Though the original vision has yet to be fully realized, commercial computer programs implementing a crude approximation to the concept were introduced recently [26, 27]. These computer programs address the complex K_m -synthesis problem via the so-called $D \Leftrightarrow F$ iteration in which one iteratively optimizes first a diagonal scaling frequency response matrix $D(j\omega)$ for a fixed control law $F(s)$ and then optimizes the control law $F(s)$ with the diagonal scaling $D(s)$ fixed. Each of the optimizations are known to be convex individually, though the combined problem is unfortunately not. Thus, even under ideal circumstances the $D \Leftrightarrow F$ iteration approach to K_m -synthesis cannot be guaranteed to be globally convergent. Nevertheless, each iteration tends to improve a bound on performance and robustness so that it can be an effective approach to robust control system design.

However, there is one major problem with the approaches to K_m -synthesis in [22, 23, 26, 27]. They all require curve fitting approximations as an intermediate step after each $D(j\omega)$ optimization, in order to obtain a rational, state-space realizable diagonal scaling matrix $D(s)$ whose frequency response approximates that of the $D(j\omega)$ computed in the $D(s)$ portion of the $D \Leftrightarrow F$ iteration. It is this curve fitting phase that is the principal

obstacle to the realization of the original vision of a completely automated K_m -synthesis procedure for robust control design.

In this paper we show how to bypass the difficult and awkward curve fitting phase of K_m -synthesis. We develop theory and associated conceptual algorithms, ready for computer implementation, that directly compute optimal fixed order diagonal scalings $D(s)$ so that curve fitting is unnecessary. We show that in fact it suffices to consider polynomial diagonal scaling matrices $D(s)$ of a fixed degree and that the resultant optimization is essentially finite dimensional and convex.

We adopt the more general, less conservative multiplier formulation of the K_m -synthesis problem introduced by us in [21]. In this formulation the usual diagonal scalings are replaced with complex diagonal multipliers acting on a positive-real, bilinearly-transformed system. Our multiplier formulation includes the diagonal scaling approach as a special case, but it also has the advantage that it is capable of producing less conservative K_m -synthesis control law designs for the case in which some or all of the uncertain gains are known to be real. As noted in [20], the multiplier stability robustness conditions for systems with mixed real/complex uncertainty offer a new perspective on, but are mathematically equivalent to, the conditions of [18, 19]. The advantages of the new multiplier perspective in paving the way for a reliable, fully-automated K_m -synthesis procedure become clear in the present paper.

The paper is organized as follows. In section III we describe our multiplier formulation of the K_m -synthesis problem. In section IV we show that, in seeking fixed order rational diagonal multipliers, we can without loss of generality restrict our attention to fixed degree diagonal polynomial multipliers. In section V we present our main theoretical results which characterize fixed order multiplier optimization as the equivalent of an m -form numerical range optimization similar to that considered by [17] — a problem which is smooth, convex and readily solved via iterative numerical algorithms. One such algorithm is given in Section VI. Conclusions are in Section VII.

II Notation

We employ the notation in Table I. Additionally, we use the notation $\text{lftf}(P_1, P_2)$ to refer to the linear fractional transformation resulting from the interconnection of two systems as shown in Figure 1. Note that in general, both systems P_1 and P_2 are two port systems, but that the linear fractional transformation is still defined if either system is a one port, i.e., if either $\dim(u_1) = \dim(y_1) = 0$ or $\dim(u_2) = \dim(y_2) = 0$. We refer to transfer function matrices in sector $[0, \infty]$ as *positive real*.

Table I: Notation.

$:=$	denotes (is defined as)
\mathcal{R}	the set of real numbers
\mathcal{C}	the set of complex numbers
\mathcal{Z}	the set of integers
\mathcal{Z}_+	the set of nonnegative integers
$\text{co}(\mathcal{X})$	the closed convex hull of a set \mathcal{X}
$\text{col}(x_1, \dots, x_n)$	the column vector $[x_1^T, x_2^T, \dots, x_n^T]^T$
$\text{col}_{i=1, \dots, n}(x_i)$	$\text{col}(x_1, \dots, x_n)$
$\text{diag}_{i=1, \dots, n}(A_i)$	the block diagonal matrix $\text{diag}(A_1, A_2, \dots, A_n)$
\bar{A}	the complex conjugate of A
A^T	the transpose of A
A^*	\bar{A}^T , if A is a matrix
$A^*(s)$	$\bar{A}^T(\bar{s})$
$\text{deg}(p(s))$	degree of the polynomial matrix $P(s)$
$\text{herm}(A(s))$	$\frac{1}{2}(A + A^*)$
$\text{asym}(A(s))$	$\frac{1}{2}(A \Leftrightarrow A^*)$
$\bar{\sigma}(A)$	The greatest singular value of the matrix A
L^2	$L^2(\Leftrightarrow \infty, \infty)$
$\ A(s)\ _{L^2}$	$= \sqrt{\int_{-\infty}^{\infty} A^*(j\omega)A(j\omega)d\omega}$
\mathcal{S}	the unit sphere in L^2 ; i.e., $\{x \in L^2 \mid \ x\ _{L^2} = 1\}$
H^∞	The set of stable transfer function matrices
$\ A(s)\ _\infty$	The H^∞ norm, $\sup_{j\omega} \sigma_{\max}(A(j\omega))$
$\text{sectf}(A(s))$	$\text{sectf}(A) = (I \Leftrightarrow A)(I + A)^{-1}$
$\text{sector}[\Leftrightarrow 1, 1]$	The set $\{G(s) \in H^\infty \mid \ G(s)\ _\infty \leq 1\}$
$\text{sector}[0, \infty]$	The set $\{G(s) \in H^\infty \mid \text{herm}(G(j\omega)) \geq 0 \forall \omega \in \mathcal{R}\}$

III K_m -Synthesis using Multipliers

One of the drawbacks associated with the complex K_m or μ synthesis is that it treats each uncertainty as being bounded by a complex disc. This can lead to conservativeness if one or more of the uncertainties Δ_i is known to be real. In [21] we briefly introduced the idea of using generalized Popov multipliers to formulate a new procedure that can find a robust \mathbf{H}^∞ controller that takes into account the real uncertainties of the system. In this section we describe our multiplier approach to mixed real/complex K_m -synthesis in greater detail.

A bilinear sector transform plays an essential role in our multiplier for-

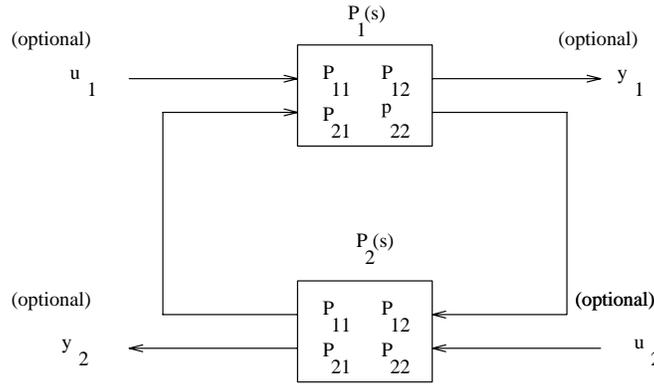


Figure 1: Linear fractional transformation

mulation of the μ -synthesis procedure. It converts the regular H^∞ problem of placing the closed loop system inside sector $[\epsilon, 1]$ to an equivalent problem of placing it inside sector $[0, \infty]$; Popov multiplier techniques are applicable to the latter problem. Multipliers which preserve the sectoricity properties of the sector-transformed block diagonal uncertainty matrix $\Delta = \text{diag}(\Delta_1, \dots, \Delta_p)$ are then chosen so as to maximize the gamma for which the transformed closed-loop system is in the sector $[0, \infty]$ meets the robust stability conditions of the positive relation stability criterion [5] which states that a sufficient condition for a negative feedback interconnection of two stable systems to be stable is that one of the systems be positive and the other to be strictly positive. A stable linear time-invariant system $G(s)$ is positive if it is positive real; it is strictly positive if for some $\epsilon > 0$, $G(s) \epsilon I$ is positive real.

The bilinear sector transform is defined as follows

Definition 1 Given a square transfer function matrix $G(s)$, the bilinear sector transform of $G(s)$ is defined to be

$$\text{sectf}(G(s)) = (I \epsilon G)(I + G)^{-1} \tag{1}$$

From the H^∞ control perspective, one key property of the sector transform is that it maps the sector $[\epsilon, 1]$ onto sector $[0, \infty]$.

Lemma 1 *Let $T = \text{sectf}(G)$. Then $G \in \text{sector}[\Leftrightarrow 1, 1]$ if and only if $T \in \text{sector}[0, \infty]$*

Proof: This is a well known fact (e.g., [28]). It may be easily proved as follows:

$$\begin{aligned}
G \in \text{sector}[\Leftrightarrow 1, 1] &\Leftrightarrow \|G\|_\infty \leq 1 \\
&\Leftrightarrow I \Leftrightarrow G^*G \geq 0 \\
&\Leftrightarrow \text{herm}((I + G)^*(I \Leftrightarrow G)) \geq 0 \\
&\Leftrightarrow \text{herm}((I \Leftrightarrow G)(I + G)^{-1}) \geq 0 \\
&\Leftrightarrow T \in \text{sector}[0, \infty]
\end{aligned} \tag{2}$$

□

Note that the relation $T = \text{sectf}(G)$ can be realized via a linear fractional transformation

$$T(s) = \text{lftf}(\text{SECTF}, G(s)) \tag{3}$$

where SECTF denotes the matrix

$$\text{SECTF} := \begin{bmatrix} I & \Leftrightarrow 2I \\ I & \Leftrightarrow I \end{bmatrix} \tag{4}$$

The sector transform sectf is equal to its own inverse; that is,

$$G = \text{sectf}(\text{sectf}(G)). \tag{5}$$

Because it is equal to its own inverse, it follows from Lemma 1 that sectf also maps $\text{sector}[0, \infty]$ onto $\text{sector}[\Leftrightarrow 1, 1]$. Thus, the sector transform sectf can be used to transform H^∞ control synthesis problems into positive real control synthesis problems and vice versa — see Figure 2.

Since the sector transform sectf is equal to its own inverse, the stability and robustness properties are unaffected by the insertion of the two sectf transformations. The transformed T and Δ matrices are given by

$$\tilde{T} = \text{lftf}(\text{SECTF}, T) = \text{sectf}(T) = (I \Leftrightarrow T)(I + T)^{-1} \tag{6}$$

$$\tilde{\Delta} = \text{lftf}(\Delta, \text{SECTF}) = \Leftrightarrow \text{sectf}(\Leftrightarrow \Delta) = \Leftrightarrow (I + \Delta)(I \Leftrightarrow \Delta)^{-1} \tag{7}$$

One readily checks that $\tilde{\Delta}$ has the same diagonal structure as Δ , viz.,

$$\tilde{\Delta} = \text{diag}(\tilde{\Delta}_1, \dots, \tilde{\Delta}_p) \tag{8}$$

where

$$\tilde{\Delta}_i = \Leftrightarrow \text{sectf}(\Leftrightarrow \Delta_i) = \Leftrightarrow (I_{k_i} + \Delta_i)(I_{k_i} \Leftrightarrow \Delta_i)^{-1} \tag{9}$$

Since by hypothesis $\Delta_i \in \text{sector}[\Leftrightarrow 1, 1]$, we also have $\Leftrightarrow \Delta_i \in \text{sector}[\Leftrightarrow 1, 1]$ and hence by Lemma 1

$$\Leftrightarrow \tilde{\Delta}_i \in \text{sector}[0, \infty]. \tag{10}$$

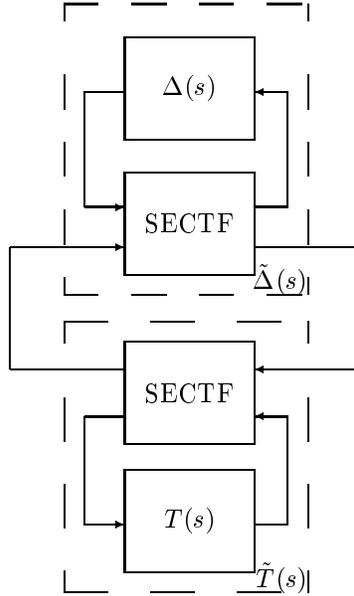


Figure 2: Transformation of H^∞ synthesis to positive-real synthesis.

Further, it is clear from (9) that $\tilde{\Delta}_i \in \mathcal{R} \cup \{\infty\}$ if and only if $\Delta_i \in \mathcal{R} \cup \{\infty\}$.

The class of multipliers which have the desired property of preserving the sectoricity of the transformed uncertainty matrix $\tilde{\Delta}$ are those in the closure of the set \mathcal{M} defined as follows:

Definition 2 *The class \mathcal{M} of multiplier matrices for the K_m -synthesis problem is defined to be the set of transfer function matrices satisfying*

$$\text{herm}(M_{k_i}(j\omega)) > 0 \quad \forall \omega \tag{11}$$

where $M(s)$ is of the diagonal form

$$M(s) = \text{diag}(m_1(s)I_{k_1}, m_2(s)I_{k_2}, \dots, m_p(s)I_{k_p}) \tag{12}$$

and the $m_i(s)$ are non-zero scalar valued transfer functions, having no poles or zeros on the $j\omega$ -axis, with real coefficients and with the additional property that, if Δ_i is a complex uncertainty then the corresponding $m_i(j\omega)$ is real; i.e.,

$$\text{Im } m_i(j\omega) = 0 \quad \forall \omega, \quad \text{if } \Delta_i \text{ is a complex uncertainty block.} \tag{13}$$

It is clear that if $M(s)$ is invertible, then $M(s) \in \mathcal{M}$ if and only if $(M(s))^{-1} \in \mathcal{M}$. While in general the members of the class \mathcal{M} are neither stable nor minimum-phase, we note *any* $M(s) \in \mathcal{M}$ can be factored as

$$M(s) = (M_2^*(s))^{-1}M_1(s) \quad (14)$$

where M_1 and M_2 and their inverses are stable and rational. We further have the following result:

Lemma 2 *Let $M_1(s)$ and $M_2(s)$ be stable transfer function matrices with stable inverses satisfying (14) and let $G(s)$ be a stable transfer function matrix. Then,*

$$\text{herm}(M(j\omega)G(j\omega)) > 0 \quad \forall \omega \quad (15)$$

if and only if

$$M_1(s)G(s)M_2(s) \in \text{sector}[0, \infty] \quad (16)$$

Proof: Multiply $M_1(s)G(s)M_2$ on left and right by $(M_2^{-1})^*$ and M_2^{-1} respectively. The result follows. \square

The implication of Lemma 2 is that one need not actually compute the stable minimum phase factors $M_1(s)$, $M_2(s)$ of $M(s)$ in order to determine their existence; rather, it suffices verify that some $M \in \mathcal{M}$ satisfies the frequency response condition (15).

The multipliers in $M \in \mathcal{M}$ have the key property that for any admissible block diagonal matrix Δ , $M_2^{-1}\Delta M_1^{-1}$ is inside $\text{sector}[0, \infty]$ if and only if Δ is inside $\text{sector}[0, \infty]$. The classical Popov multiplier $1+qs$ is in the class \mathcal{M} ; we therefore refer the multipliers $M \in \mathcal{M}$ as *generalized Popov multipliers*. (Of course, if we wanted to consider nonlinear Δ_i 's as did Popov, then it would be necessary to impose other more complicated restrictions as in [29, 30] in our Definition 2 of \mathcal{M} .)

The multiplier formulation of the mixed real/complex uncertainty K_m -synthesis problem formulation goes as follows (See Figure 3)

$$\max \gamma \quad (17)$$

subject to the constraints that the controller $F(s)$ be stabilizing and

$$(M_2^*)^{-1}M_1 \in \mathcal{M}$$

$$\|T_{\bar{y}_1 \bar{u}_1}\|_\infty \leq 1.$$

In other words, we want to find the greatest real number γ such that for some generalized Popov multiplier M the infinity norm of the cost function $T_{\bar{y}_1 \bar{u}_1}$ is less than or equal to one. For such a controller, the corresponding γ will be a lower bound on the size of the smallest destabilizing real uncertainties.

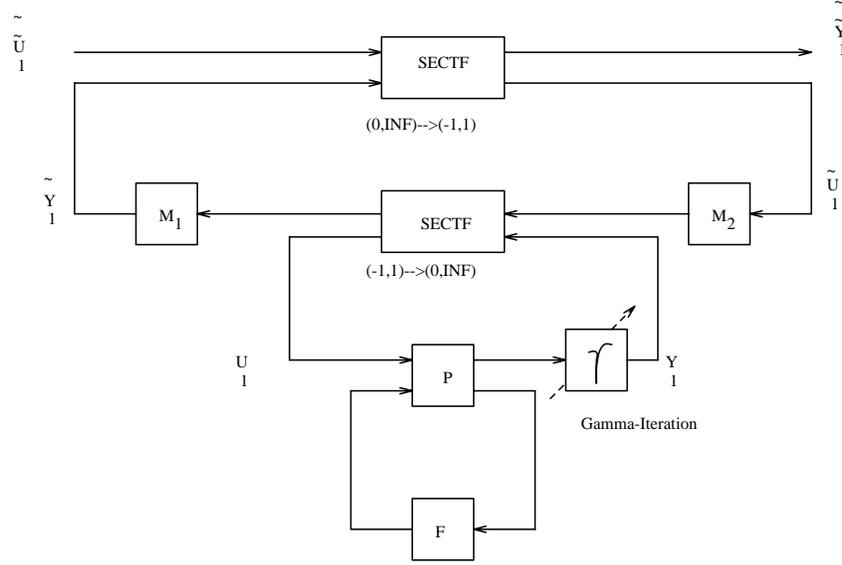


Figure 3: K_m -Synthesis with Multipliers

Our multiplier-based mixed real/complex K_m -synthesis design procedure is summarized by the following conceptual algorithm, essentially a straightforward adaptation to the multiplier context of the $D \Leftrightarrow F$ iteration of the traditional diagonal scaling approach [22, 23]. The algorithm returns the K_m -synthesis control law $F_{best}(s)$ along with the corresponding multiplier $M_{best} \in \mathcal{M}$ and cost $\gamma_{best} := 1/\sup_{\omega} \mu(T_{y_1 u_1}(j\omega))$. The algorithm involves an “M-F iteration” in which one iteratively alternates between (i) fixing the multiplier $M(s)$ and solving for an H^∞ optimal control law $F(s)$ and (ii) fixing the controller $F(s)$ and optimizing the multiplier $M(s) \in \mathcal{M}$. The algorithm terminates when neither (i) nor (ii) results in any further improvement in γ_{best} . See Figure 3.

Algorithm 1 (Multiplier μ -Synthesis)

Step 1 Initialize by solving the conventional H^∞ optimal control problem of finding a stabilizing H^∞ controller $F(s)$ which maximizes the value of γ for which

$$\|T_{y_1 u_1}\|_\infty \leq 1.$$

Set $M_{best}(s) = M_1(s) = M_2(s) = I$, $F_{best}(s) = F(s)$, $\gamma_{best} = \gamma_{oldbest} = \gamma$. (Note that, for this value of γ , $T_{y_1 \tilde{u}_1}$ is inside sector $[\Leftrightarrow 1, 1]$ and $T_{y_1 \tilde{u}_1}$ is inside sector $[0, \infty]$.)

Step 2 Iteratively increase γ and solve the convex optimization problem of computing an improved $M(s) \in \mathcal{M}$ so as to maximize

$$\rho_M = \min_{\|x\|=1} x^* (\text{herm}(M(j\omega)T_{\bar{y}_1\bar{u}_1}(j\omega)))x. \quad (18)$$

If $\rho_M > 0$ set $\gamma_{best} = \gamma$, $M_{best}(s) = M(s)$ and repeat Step 2; otherwise, continue to Step 3.

Step 3 Compute the factorization $M_{best} = (M_2^*)^{-1}M_1$ and augment the plant as shown in Figure 3.

Step 4 Increase γ and solve the H^∞ optimal control problem finding a stabilizing controller $F(s)$ which minimizes the cost

$$\rho_F = \min_{\text{stabilizing } F(s)} \|T_{\bar{y}_1\bar{u}_1}\|_\infty.$$

If $\rho_F \leq 1$ set $F_{best}(s) = F(s)$ and repeat Step 4; otherwise continue to Step 5.

Step 5 If $\gamma_{oldbest} < \gamma_{best}$, set $\gamma_{oldbest} = \gamma_{best}$ and goto Step 2; otherwise stop.

□

Comments:

- (i) A standard “ γ -iteration” binary search would be an appropriate method for iteratively adjusting γ as required in Steps 2 and 4 of the foregoing conceptual algorithm.
- (ii) There is a close relation between the multiplier approach to μ -synthesis and the diagonal scalings used in the traditional complex μ -synthesis. If $D(s)$ is a stable minimum-phase invertible diagonal matrix such that $\bar{\sigma}(DTD^{-1}) < 1$, then it may be easily shown that $M(s) = D^*(s)D(s)$ is in the class \mathcal{M} and satisfies $\text{herm}(M(s) \text{sectf}(T(s))) > 0$; in this case one may take $M_1(s) = D(s)$ and $M_2(s) = D^{-1}(s)$. Moreover, if only complex uncertainties Δ_i are present, then every multiplier $M \in \mathcal{M}$ has the property that it is positive and real for all frequency $j\omega$ and hence factorizable as $M(s) = D^*(s)D(s)$ for some stable minimum-phase $D(s)$. It follows that diagonal scaling conditions of complex μ -synthesis may be regarded as equivalent to our multiplier conditions; of course, when some of the uncertainties are real our multiplier approach is more powerful (i.e., less conservative).

- (iii) A subtle, but critically important problem arises in the above algorithm because the optimization of $M(s)$ in Step 2 is most naturally addressed pointwise at each, leading to an irrational frequency response representation of $M(j\omega)$ whereas the H^∞ optimization in Step 4 requires finite-dimensional state-space realization for $M(s)$. Indeed, since the order of the H^∞ control law $F(s)$ that is computed in Step 4 is generically equal to one less than the order $M_1(s)T_{\tilde{y}_1\tilde{u}_1}M_2(s)$, it is desirable that the order of $M(s)$ be fixed at some reasonably small value lest the control law $F(s)$ be too complicated to implement. Thus, in Step 2 it is critical that the multiplier optimization be subject to an order constraint. One might handle this first ignoring the order constraint and computing the optimal $M(j\omega)$ at each frequency ω , then doing a rational or polynomial curve fit to find a finite order approximation to $M(j\omega)$. This is analogous to the curve fitting that is used with the diagonal-scaling frequency responses $D(j\omega)$ that arise in the conventional approach to the complex K_m -synthesis problem [22, 23, 26, 27]. However, the use of such curve fitting is far from satisfactory since there is no a priori way to assess the tradeoff between the accuracy of the curve fit and the resultant degradation in achievable performance γ . Fortunately, as we will show, the affine nature of the multiplier optimization problem makes it practical to directly impose an order constraint on $M(s)$ in performing the multiplier optimization in Step 2, thereby bypassing the awkward and difficult curve fitting step associated with the conventional approaches to K_m -synthesis.

IV Fixed-Order and Polynomial Multipliers

As we have noted, a critical step in the multiplier formulation of the K_m -synthesis problem is the computation of the optimal $M(s)$ subject to an order constraint. Mathematically, the problem may be formulated as follows: Given a transfer function $T(s)$ and a nonnegative integer p , to find an order p rational, biproper diagonal multiplier matrix $M(s) = \text{diag}\{m_i(s)I_{k_i}\} \in \mathbf{M}$ for which the matrix $\text{herm}(M(s)T(s))$ is nonnegative definite. That is, we wish to find an $M(s) \in \mathcal{M}$ such that

$$\text{herm}(M(j\omega)T(j\omega)) > 0 \quad \forall \omega. \quad (19)$$

Equivalently, we wish to find $M(s) \in \mathcal{M}$ such that

$$x^* \text{herm}(M(j\omega)T(j\omega))x > 0 \quad \forall \omega, \forall \|x\| = 1. \quad (20)$$

The weak point in the conventional approach to μ -synthesis is this step. In the conventional approach to complex K_m -synthesis [22, 23, 26, 27] one

computes an optimal diagonal scaling, say $D(j\omega)$, pointwise at each frequency ω . One then does an *ad hoc* curve fit in order to find a low-order rational approximation $D(s)$ to the optimal $D(j\omega)$. Alas, the sensitivity of $\mu(T_{y_1 u_1})$ to the resultant inaccuracy in $D(s)$ cannot be determined a priori and there is no single good measure of what constitutes a good approximation.

One could employ essentially the same curve fitting approach in the multiplier case, but this would likely lead to the same sensitivity problems that beleaguer curve fitting in the diagonal scaling approach to K_m -synthesis. Indeed, the sensitivity of $\mu(\text{sectf}(M(s)\text{sectf}(\gamma T_{y_1 u_1}(s))))$ to variations in $M(s)$ will in general be very difficult to assess a priori. Moreover, the problem of finding a suitable low-order rational approximation to $M(s)$ is compounded by the constraint that the approximant must be in \mathcal{M} . Thus, it would be much preferable if curve fitting could be bypassed altogether and instead the order constraint were explicitly imposed on the $M(s)$ optimization of Step 2 of Algorithm 1. As we shall show, this is easy to do in the multiplier framework.

The following lemma establishes that in searching for *rational* multipliers satisfying (19) we may without loss of generality confine our attention to real *polynomial* multipliers; i.e., elements of \mathcal{M} for which the diagonal multiplier transfer functions $m_i(s)$ are of the forms

$$m_i(s) = \sum_{j=0}^n m_{ij} s^j, \quad \text{if } \Delta_i \in \mathcal{R} \quad (21)$$

$$m_i(s) = \sum_{j=0}^n m_{ij} s^{2j}, \quad \text{if } \Delta_i \in \mathcal{C}. \quad (22)$$

Lemma 3 *Equation (19) is satisfied for some real rational $M(s) \in \mathcal{M}$ if and only if there exists a real polynomial $M(s) \in \mathcal{M}$ for which (19) holds. Moreover, if such a rational $M(s)$ is factored as $M(s) = \frac{1}{d(s)}N(s)$ where $N(s)$ is a polynomial matrix and $d(s)$ is scalar valued polynomial, then the degree of the corresponding polynomial matrix $M(s)$ need not be greater than the sum of the degrees of $d(s)$ and $N(s)$.*

Proof: First note that for any $n \times n$ invertible matrix S we have

$$T > 0 \Leftrightarrow STS^* > 0. \quad (23)$$

Also note that any rational $M(s) \in \mathcal{M}$ may be written as $\frac{1}{d(s)}N(s)$. Taking $S = d(s)I$ in (23), it follows that

$$\text{herm}(MT) > 0 \Leftrightarrow \text{herm}(\tilde{M}T) > 0 \quad (24)$$

where $\tilde{M}(s) = d^*(s)d(s)M(s) = d^*N(s) \in \mathcal{M}$ is a diagonal polynomial matrix of degree equal to the sum of the degrees of $N(s)$ and $d(s)$. \square

Comments:

- (i) Of course, in some situations it is preferable to work with strictly proper transfer functions $M(s)$ which, unlike polynomials, may be realized in state-space form. Thus, if one has computed a polynomial $M(s)$, it is always possible to determine an equivalent proper multiplier by performing the reverse of the operations in the proof of Lemma 3. That is, one may always substitute for $M(s)$ the equivalent proper multiplier

$$\tilde{M}(s) = \frac{1}{d^*(s)d(s)} M(s)$$

where $d(s)$ is *any* scalar polynomial of degree at least half that of $M(s)$.

- (ii) Since polynomials are linear in their coefficients and Lemma 3 implies that in our search for a fixed order $M(s)$ we may restrict our attention to polynomials. The implication is that the order-constrained multiplier optimization that one would like to solve in Step 2 of Algorithm 1 may be directly treated as a convex optimization over the coefficients of the polynomials $m_i(s)$, ($i = 1, \dots, p$). That is, we may bypass the need for curve fitting without losing convexity or otherwise complicating the problem in any significant way.
- (iii) One may readily generalize Lemma 3 by substituting polynomials in functions of s for polynomials in s . For example, it would also suffice to consider, say, $m_i(s)$ polynomials in $(as + b)/(cs + b)$ where a, b, c, d are given scalars. This may have advantages when one prefers to have $M(s)$ be proper so as to be state-space realizable.

V Main Results

The above results establish that the fixed order multiplier optimization may be treated by optimizing the cost (19) over the set of $M(s) = \text{diag}\{m_i(s)I_{k_i}\} \in \mathcal{M}$ for which the SISO transfer functions $m_i(s)$ are polynomials of a fixed degree with real coefficients; that is,

$$m_i(s) = \sum_{j=0}^{\text{deg}(m_i(s))} m_{ij}s^j, \quad \text{if } \Delta_i \in \mathcal{R} \tag{25}$$

and

$$m_i(s) = \sum_{j=0}^{\deg(m_i(s))/2} m_{ij} s^{2j}, \quad \text{if } \Delta_i \in \mathcal{C} \quad (26)$$

where $m_{ij} \in \mathcal{R}$. Note that the polynomials (26) corresponding to complex Δ_i are restricted to be have even degree and to have only even powers of s ; this is necessary and sufficient to ensure that $m_i(j\omega) \in \mathcal{R} \forall \omega$ for each complex Δ_i as required by Definition 2. Thus the number, say r_i , of coefficients m_{ij} in the i -th multiplier $m_i(s)$ is

$$r_i = \begin{cases} 1 + \deg(m_i(s)), & \text{if } \Delta_i \in \mathcal{R} \\ 1 + \deg(m_i(s))/2, & \text{if } \Delta_i \in \mathcal{C}. \end{cases} \quad (27)$$

Notice that an $M(s)$ satisfying (25,26) is an element of \mathcal{M} if and only if it satisfies the condition $\text{herm}(M(j\omega)) > 0 \forall \omega$ — this is a condition of the same form as (19), except with $T(j\omega) = I$. Since for any two hermitian matrices A, B , it holds that $\text{diag}(A, B) > 0$ if and only if both $A > 0$ and $B > 0$, we have the following result.

Theorem 1 *Let $\tilde{\mathcal{M}}$ be a subset of \mathcal{M} , let $T(s)$ be transfer function matrix, and let*

$$Q(s, M(s)) := \text{herm}(\text{diag}(M(s), M(s)T(s))). \quad (28)$$

Then there exists a multiplier $M(s) \in \tilde{\mathcal{M}}$ satisfying (19) if and only if

$$\rho := \inf_{x \in \mathcal{S}} \int_{-\infty}^{\infty} x^*(j\omega) Q(j\omega, M(j\omega)) x(j\omega) d\omega > 0 \quad (29)$$

□

In what follows we shall restrict our attention to a particular subset $\tilde{\mathcal{M}}$ of \mathcal{M} .

Definition 3 *$\tilde{\mathcal{M}}$ is the set of all polynomial $M \in \mathcal{M}$ satisfying (25,26) hold.*

□

Suppose $M \in \tilde{\mathcal{M}}$. Define $z \in \mathcal{R}^{n_z}$ as

$$z := \underset{i=1, \dots, p}{\text{col}} \underset{j=0, \dots, (r_i-1)}{\text{col}} \{m_{ij}\} \quad (30)$$

where

$$n_z := \sum_{i=1}^p r_i. \quad (31)$$

Thus, $z \in \mathcal{R}^{n_z}$ is a column vector whose elements are the m_{ij} 's. Clearly, for any $M \in \tilde{\mathcal{M}}$ we have that $Q(s, M(s))$ may be additively decomposed as

$$Q(s, M(s)) := \sum_{i=1}^{n_z} z_i Q_i(s) \tag{32}$$

where $Q_i(s)$ are hermitian matrices that do not depend on the coefficients m_{ij} .

We further define the function

$$f(x) := \underset{i=1, \dots, n_z}{\text{col}} \int_{-\infty}^{\infty} x^*(j\omega) Q_i(j\omega) x(j\omega) d\omega \tag{33}$$

With this definition, the argument of equation (29) may be written equivalently as $z^T f(x)$; that is,

$$z^T f(x) \equiv \int_{-\infty}^{\infty} x^*(j\omega) Q(j\omega, M(j\omega)) x(j\omega) d\omega. \tag{34}$$

This leads to our main result, a corollary to Theorem 1 characterizing the computation of the optimal fixed order polynomial multiplier matrices satisfying (19) in terms of a sort of numerical range optimization problem — a problem for which there are known and reliable solution approaches (e.g., [17, 20].)

Corollary 1 (Key Result) *Let $\tilde{\mathcal{M}}$ be a subset of \mathcal{M} and let $T(s)$ be transfer function matrix. Then there exists a multiplier $M(s) \in \tilde{\mathcal{M}}$ satisfying (19) if and only if*

$$0 \notin \text{co}(f(\mathcal{S})) \tag{35}$$

Proof: From Theorem 1, the existence of a multiplier in $\tilde{\mathcal{M}}$ satisfying (19) is equivalent to existence of a $z \in \mathcal{R}^{n_z}$ such that

$$\inf_{x \in \mathcal{S}} z^T f(x) > 0 \tag{36}$$

which is equivalent to (35). □

VI Algorithm and Convergence

Corollary 1 establishes that the problem of finding a polynomial $M(s) \in \tilde{\mathcal{M}}$ such that (19) holds is equivalent to the problem of determining whether or not the point $0 \in \mathcal{R}^{n_z}$ is in the convex hull of the set $f(\mathcal{S}) \in \mathcal{R}^{n_z}$. This condition may be tested via the following algorithm. Since the condition

$0 \notin \text{co}(f(\mathcal{S}))$ is equivalent to the existence of a hyperplane separating $f(\mathcal{S})$ from 0; i.e., it is equivalent to the existence of a $z \in \mathcal{R}^{n_z}$ such that (36) holds. Any such z gives us the desired multiplier $M(s) \in \tilde{\mathcal{M}}$, z being simply the vector (30) containing the coefficients m_{ij} of the polynomials $m_i(s)$ on the diagonal of the desired polynomial matrix $M(s) \in \tilde{\mathcal{M}}$. Keeping the foregoing in mind, we are led to our main result, the following conceptual algorithm for a computing a multiplier for which (19) holds.

Algorithm 2 (Fixed-Order Multiplier Optimization)

Step 1 Initialize by choosing a point $z^{(1)} \in \text{co}(f(\mathcal{S}))$ and setting $k=1$.

Step 2 Fix $z = z^{(k)}$ and set the coefficients m_{ij} of $M(s)$ equal to the values determined by z . Solve the infimization

$$\inf_{x \in \mathcal{S}} \int_{-\infty}^{\infty} x^*(j\omega)Q(s, M(s))x(j\omega) d\omega \quad (37)$$

and let $f^{(k)}$ be the minimizing value of $f(x)$.¹

Step 3 Compute the minimum norm element in $\text{co}\{f^{(k)}, z^{(k)}\}$:

$$z^{(k+1)} = z^{(k)} + \min\{\alpha, 1\}(f^{(k)} \Leftrightarrow z^{(k)}) \quad (38)$$

where

$$\alpha = \frac{\Leftrightarrow(f^{(k)} \Leftrightarrow z^{(k)})^T z^{(k)}}{(f^{(k)} \Leftrightarrow z^{(k)})^T (f^{(k)} \Leftrightarrow z^{(k)})} \quad (39)$$

Increment k by one and return to Step 2. □

The following theorem establishes that Algorithm 2 computes a multiplier satisfying (19) in a finite number of steps whenever one exists. The key to the proof is the observation that, as $k \rightarrow \infty$, the sequence $\{z^{(k)}\}$ converges to the unique minimum norm element of $\text{co}(f(\mathcal{S}))$.

Theorem 2 (Convergence) *If there exists a multiplier $M(s) \in \tilde{\mathcal{M}}$ such that (19) holds then Algorithm 2 converges as $k \rightarrow \infty$ to a corresponding $z \in \mathcal{R}^{n_z}$; otherwise it converges to $z = 0$.*

Proof: The result is essentially an application of Theorem 2 of [31]. Note that the Algorithm 2 implicitly defines a mapping, say $g : \mathcal{R}^{n_z} \rightarrow \mathcal{R}^{n_z}$ which maps $z^{(k)}$ into $z^{(k+1)}$, i.e.,

$$z^{(k+1)} = g(z^{(k)}). \quad (40)$$

¹Note that if the infimum (37) is positive, then $M(s)$ is a multiplier satisfying (19); so one may chose to stop the algorithm at this point if an “optimal” (i.e., minimum norm) z is not needed.

Further, the mapping $g : \mathcal{R}^{n_z} \rightarrow \mathcal{R}^{n_z}$ maps the compact set $\text{co}(f(\mathcal{S}))$ into itself and is contractive in the sense that, for some continuous function $\phi : \mathcal{R} \rightarrow \mathcal{R}$ with $\phi(0) = 0$ and $\phi(\epsilon) > 0 \forall \epsilon > 0$,

$$\|g(z)\| \leq (1 \Leftrightarrow \phi(\|z \Leftrightarrow z_{opt}\|)) \|z\| \quad (41)$$

where z_{opt} denotes the unique minimum norm element of the closed, convex, compact set $\text{co}(f(\mathcal{S})) \subset \mathcal{R}^{n_z}$. It follows that at each iteration $\|z^{(k+1)}\| \leq \|z^{(k)}\|$ with equality holding if and only if $z^{(k)} = z_{opt}$ and that the sequence $\{z^{(k)}\}$ is globally convergent to z_{opt} — see [31] for further details. Further, since z_{opt} is the minimum norm element of $\text{co}(f(\mathcal{S}))$, it follows that if $z_{opt} \neq 0$, then $z_{opt}^T f(x) > 0 \forall x \in \mathcal{S}$ and the coefficients m_{ij} of the corresponding multiplier are the elements of the vector $z \in \mathcal{R}^{n_z}$ — see (30,32). Otherwise, the unique minimum norm element of $\text{co}(f(\mathcal{S}))$ is $z_{opt} = 0$. The result follows from Corollary 1. \square

The question remains, how can one perform the optimization over x required in Step 2 of Algorithm 2? It turns out that this is actually quite easy as a consequence of the following variant of the Positive Real Lemma [32].

Lemma 4 *Consider the transfer function $G(s) = C(Is \Leftrightarrow A)^{-1}B + D$. Suppose the matrix $\text{herm}(D) \Leftrightarrow \gamma I$ is invertible and that the pair $\{A, B\}$ is stabilizable. Then $G(j\omega)$ satisfies*

$$\text{herm}(G(j\omega)) > \gamma I \quad \forall \omega \quad (42)$$

if and only if the hamiltonian system matrix

$$H = \begin{bmatrix} A \Leftrightarrow BR^{-1}C & \Leftrightarrow BR^{-1}B^* \\ C^*R^{-1}C & \Leftrightarrow (A \Leftrightarrow BR^{-1}C)^* \end{bmatrix} \quad (43)$$

where

$$R = D + D^* \Leftrightarrow 2\gamma I \quad (44)$$

has no eigenvalues on the imaginary axis of the complex plane. Furthermore, if γ_0 is the least γ for which (42) fails to hold or, equivalently, for which (43) has a $j\omega$ -axis eigenvalue at, say, $j\omega_0$ then

$$\inf_{x \in \mathcal{S}} \int_{-\infty}^{\infty} x^*(j\omega) \text{herm}(G(j\omega) \Leftrightarrow \gamma_0 I) x(j\omega) d\omega = 0 \quad (45)$$

and an infimizing sequence $\{x^{(k)}\}_{k=0}^{\infty}$ is

$$x(j\omega) = x_0 u^{(k)}(\omega \Leftrightarrow \omega_0) \quad (46)$$

where $\{u^{(k)}\}_{k=0}^{\infty}$ is any sequence in \mathcal{S} such that $|u^{(k)}(j\omega)|^2$ converges to the Dirac delta function and x_0 is any unit vector in the nullspace of $\text{herm}(\gamma_0 I \Leftrightarrow G(j\omega_0))$, i.e., any vector satisfying

$$\text{herm}(\gamma_0 I \Leftrightarrow G(j\omega_0))x_0 = 0. \quad (47)$$

Proof: We only give a brief sketch. Clearly $\text{herm}(G(j\omega)) > \gamma$ holds if and only if $G(j\omega) + G(j\omega)^* \Leftrightarrow 2\gamma I = T^*(s)T(s)$ for some stable, minimum-phase $T(s)$, i.e., if and only if $G(j\omega) + G(j\omega)^* \Leftrightarrow 2\gamma I$ has a spectral factor $T(s)$ with all its poles and zeros strictly in the left half of the complex plane \mathcal{C} . But, under the assumptions of the lemma, this is equivalent to the solvability of the linear quadratic state-feedback optimal control problem of minimizing the cost

$$J = \int_0^{\infty} \begin{bmatrix} x^* & u^* \end{bmatrix} \begin{bmatrix} 0 & C^* \\ C & D + D^* \Leftrightarrow 2\gamma I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} dt. \quad (48)$$

for the plant

$$\dot{x} = Ax + Bu. \quad (49)$$

and this linear quadratic state-feedback problem has a solution if and only if the corresponding Hamiltonian system matrix has no imaginary-axis eigenvalues. Note that in our case the cost (48) has (43) as its associated Hamiltonian system matrix. That x_0 and ω_0 exist satisfying (47) is readily seen by observing that since $\gamma = \gamma_0$ there is a frequency ω_0 at which $\text{herm}(G(j\omega))$ is positive *semi*-definite, but not positive definite. That the sequence $\{u^{(k)}\}_{k=0}^{\infty}$ is infimizing may be seen by observing that, if $x(j\omega)$ is taken to be as in (46), then (47) implies that the limiting value as $k \rightarrow \infty$ of the integral (45) is zero. \square

Of course, the foregoing Lemma is only useful if the transfer function $G(s)$ is state-space realizable in the form $G(s) = C(Is \Leftrightarrow A)^{-1}B + D$. Fortunately, even for polynomial multipliers an invertible matrix $S(s)$ can always be chosen so that even if

$$M(s)G(s) \quad (50)$$

does not have a state space realization of the form required by Lemma 4, the matrix

$$S(s)M(s)G(s)S^*(s) \quad (51)$$

does. Noting that for any invertible $S(s)$, herm of (50) is positive semi-definite if and only if herm of (51) is too, it suffices to apply Lemma 4 to the transformed system $S(s)M(s)G(s)S^*(s)$ in order to determine whether $\text{herm} M(j\omega)S(j\omega) > 0 \forall \omega$. For example, in the case of polynomial $M(s)$ and proper $G(s)$ one may take $S(s) = M^{-1}(s)$ to obtain a proper $G(s)M^{-1}(s)$.

VII Conclusions

The weak point of traditional approaches to K_m -synthesis or μ -synthesis has been the curve fitting step used to fit frequency responses of diagonal scaling functions with approximate rational realizations. We have described a conceptual algorithm, ready for implementation on a digital computer, which enables one to directly compute optimal fixed-order rational diagonal scalings and, more generally, optimal fixed-order multipliers for mixed real/complex K_m -synthesis. Our main result is Algorithm 2. The algorithm is sufficiently flexible to handle both the traditional diagonal scaling approach used for complex-uncertainty and the more general multiplier approach introduced in [21] for mixed real/complex uncertainty. The algorithm reduces the problem of finding optimal fixed-order diagonal scalings and, more generally, optimal fixed-order diagonal multiplier matrices, to the finite dimensional convex optimization problem of determining if $0 \in \text{co}(f(\mathcal{S})) \subset \mathcal{R}^{nz}$. We have proved that the algorithm is globally convergent to the unique minimum norm element of $\text{co}(f(\mathcal{S}))$, that a multiplier exists if and only if this element is nonzero. Algorithm 2 is constructive in that, whenever a multiplier exists which satisfies (19), the algorithm always produces such a multiplier in a finite number of steps. Our results enable one to bypass the awkward and imprecise curve-fitting step in K_m -synthesis/ μ -synthesis, making the overall synthesis procedure practical and reliable.

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