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THE DIAMETER OF AN INTERSECTION OF ELLIPSOIDS AND BMI ROBUST SYNTHESIS

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Abstract. It is shown that collections of bilinear matrix inequalities (BMI's) have a non-empty solution set if and only if the diameter of a certain convex set is greater than two. The convex set in question is simply the intersection of ellipsoids centered at the origin. This gives a new and simple geometric characterization of the very broad class of robust control synthesis problems that are representable in in terms of BMI's, including H^∞problems, μ/k_m-synthesis, gain-scheduling, and so forth.

Key Words. Robust control; nonlinear programming, matrix inequalities; linear systems; optimization; ellipsoids

1. INTRODUCTION

The Bilinear Matrix Inequality (BMI) has been introduced by (Safonov *et al.*, 1994; Goh *et al.*, 1994) as a geometric reformulation of many problems in robust control. The BMI is a generalization the Linear Matrix Inequality (LMI) approach to control synthesis that has been employed by various authors — see, for example, (Stoorvogel and Trentelman, 1990; Packard *et al.*, 1992; Boyd *et al.*, 1993; Packard *et al.*, 1993; Becker *et al.*, 1993; Iwasaki and Skelton, 1993). As was shown in (Safonov *et al.*, 1994), the basic form of the BMI problem is:

Problem 1 (Bilinear Matrix Inequality — BMI) Given H_{ij} $(i = 1, \ldots, n; j = 1, \ldots, m)$ symmetric matrices of dimension $k \times k$, find $x \in \mathcal{R}^n$,$y \in \mathcal{R}^m$ such that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} x_i y_i H_{ij} < 0. \tag{1}$$

■

It is possible (Safonov *et al.*, 1994) with the aid of suitably chosen "sector transforms" to formulate many (perhaps even most) of the problems considered in the robust control literature as BMI's. Such problems include fixed-order H^∞ control, μ/k_m-synthesis, decentralized control, robust gain-scheduling, simultaneous stabilization, and arbitrary combinations of these.

Thus, the solution of the BMI (1) may be regarded as the central problem in robust control. If a reliable method can be devised to solve general BMI problems, then all of the aforementioned robust control synthesis problems will be solved.

For example, an immediate consequence of the positive real lemma (e.g., Anderson and Vongpanitlerd, 1973) is that the problem of synthesizing a constant output feedback $u_2 = F y_2$ for the plant

$$\begin{bmatrix} \dot{x} \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ u_1 \\ u_2 \end{bmatrix} \tag{2}$$

so that the closed-loop transfer function $T_{y_1 u_1}(s)$ is positive real is equivalent to the problem of finding scalars p_0,f_0 and matrices $\tilde{P}$, $\tilde{F}$ such that the following matrix inequality is satisfied:

$$\text{herm}\left(\begin{bmatrix} f_0 \tilde{Z} & 0 \\ 0 & \tilde{Z}(f_0 R + U \tilde{F} V) \end{bmatrix} \right) > 0 \tag{3}$$

subject to

$$\tilde{Z} = \begin{bmatrix} \tilde{P} & 0 \\ 0 & p_0 I \end{bmatrix}; \quad \tilde{P} = \tilde{P}^T \tag{4}$$

where (cf. Packard *et al.*, 1992)

$$\tilde{F} \triangleq f_0 F (I - D_{22} F)^{-1}; \quad \tilde{P} \triangleq p_0 P \tag{5}$$

$$R \triangleq \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} \tag{6}$$

$$U \triangleq \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix} \tag{7}$$

and

$$V \triangleq \left[\begin{array}{cc} C_2 & D_{21} \end{array} \right]. \tag{8}$$

Evidently, condition (3) is linear in $p_0, \tilde{P}$, linear in $f_0, \tilde{F}$ and hence bilinear in $p_0, f_0, \tilde{P}, \tilde{F}$. That is, condition (3) is a BMI and can be represented in the form (1).

We note that the approach of Packard (1992) could be used to reduce the positive real synthesis BMI to an LMI if the controller's order were allowed to be the same as the plant's. Alternatively, the related techniques of Iwasaki and Skelton (1993) could be employed to enable positive real synthesis to be reduced to a pair of LMI's with a nonlinear coupling condition, but this technique is not applicable to all BMI's and, in any case, the nonlinear coupling seems to produce complications which make solutions difficult to compute. The attraction of the BMI representation of robust control problems is its simplicity and its generality. The ellipsoidal intersection interpretation of the BMI which we will describe is an even further simplification, without any loss of generality.

The purpose of the present paper is to show that the study of the BMI problem is equivalent to studying whether the diameter of a certain convex set is greater than two. The convex set in our case turns out to be simply the intersection of ellipsoids centered at the origin in $\mathcal{R}^{n+m}$; so the diameter is precisely twice the radius.

A conceptual algorithm for addressing the BMI problem in this context is also presented. It should be pointed out that besides its simplicity, the problem of finding the diameter of a convex set is a nonconvex programming problem, since it amounts to *maximizing* a convex function subject to a convex constraint set, a fact that attests to the difficulty of the very broad class of robust control problems represented by the BMI.

In the following we shall employ the notation $\lambda_{max}(H)$ to denote the greatest eigenvalue of a hermitian matrix H and the notation $\sigma_{max}(G) \triangleq \sqrt{\lambda_{max}(G^* G)}$ to denote the greatest singular value of a matrix G.

2. DIAMETER OF ELLIPSOID INTERSECTIONS

In this section we present our main result — a lemma which establishes equivalence of solving the BMI with the problem of computing the diameter of the intersection of ellipsoids.

We began by noting that the BMI (1) is equivalent to

$$\max_{\substack{\|z\|=1 \\ z \in \mathcal{R}^k}} x^T G(z) y < 0 \tag{9}$$

where

$$[G(z)]_{i,j} = z^T H_{ij} z \in \mathcal{R}^{n \times m}. \tag{10}$$

Let ρ be a real positive number such that

$$\rho > \max_{\|z\|=1} \bar{\sigma}(G(z)) \tag{11}$$

and let $\mathbf{C} \subset \mathcal{R}^{n+m}$ be the convex set

$$\mathbf{C} \triangleq \{ w \mid w^T Q(z) w \leq 1, \; z \in \mathcal{R}^k, \|z\| = 1 \} \tag{12}$$

where

$$Q(z) = \left[\begin{array}{cc} I & \frac{1}{\rho} G(z) \\ \frac{1}{\rho} G^T(z) & I \end{array} \right]. \tag{13}$$

Notice that, by (11), the matrix $Q(z) > 0$ for all $\|z\| = 1$. It follows that the set $\mathbf{C}$ is the intersection of ellipsoids in $\mathcal{R}^{n+m}$.

Our main result is the following lemma:

Lemma 1 (BMI Diameter) There exists a pair (x, y) such that the BMI (1) holds if, and only if, the diameter of $\mathbf{C}$ is strictly greater than 2; i.e.,

$$2 < \max_{w_1, w_2 \in \mathbf{C}} \|w_1 - w_2\|. \tag{14}$$

Proof: ($\Leftarrow$) Suppose that $\mathbf{C}$ has diameter strictly greater than 2. $\mathbf{C}$ is the intersection of an infinite number of ellipsoids and thus its maximum diameter is achieved at some $\bar{w}_1 = (\bar{x}, \bar{y})$ in the boundary of $\mathbf{C}$ and, by symmetry, also at $\bar{w}_2 = -\bar{w}_1$. It thus holds that

$$\bar{w}_1^T Q(z) \bar{w}_1 \leq 1, \quad \forall z, \; \|z\| = 1 \tag{15}$$

where

$$\bar{w}_1 \triangleq \left[\begin{array}{c} \bar{x} \\ \bar{y} \end{array} \right] \tag{16}$$

and

$$\|\bar{w}_1\| = \|\bar{x}\|^2 + \|\bar{y}\|^2 > 1. \tag{17}$$

Inequality (15) is equivalent to

$$\|\bar{x}\|^2 + \|\bar{y}\|^2 + \frac{2}{\rho} \bar{x}^T G(z) \bar{y} \leq 1, \; \forall z, \; \|z\| = 1 \tag{18}$$

from which it follows that

$$\frac{2}{\rho} \bar{x}^T G(z) \bar{y} \leq 1 - (\|\bar{x}\|^2 + \|\bar{y}\|^2) < 0. \tag{19}$$

Thus

$$\bar{x}^T G(z)\bar{y} < 0, \text{ uniformly in } z, \ \|z\| = 1. \quad (20)$$

and consequently $\bar{x}, \bar{y}$ satisfy (1).

($\Rightarrow$) Suppose $(\bar{x}, \bar{y})$ satisfy (1). Without loss of generality we may assume $\|\bar{x}\|^2 + \|\bar{y}\|^2 = 1$. It then holds that

$$\begin{aligned} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^T Q \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} &= \|\bar{x}\|^2 + \|\bar{y}\|^2 + \frac{2}{\rho}\bar{x}^T G(z)\bar{y} \quad (21) \\ &= 1 + \frac{2}{\rho}\bar{x}^T G(z)\bar{y} \quad (22) \\ &< 1 \quad (23) \end{aligned}$$

uniformly in z, $\|z\| = 1$. Thus the radius of the set $\mathbf{C}$ is strictly greater than $(\|\bar{x}\|^2 + \|\bar{y}\|^2) = 1$. Since $\mathbf{C}$ is hermitian and centered about the origin, the diameter is precisely twice the radius. Hence, the diameter of $\mathbf{C}$ is strictly greater than 2. ∎

3. DISCUSSION

The ρ's used in (13) may all be the same or they may be chosen to depend on z. All we need is to guarantee that

$$Q(z) = \begin{bmatrix} I & \frac{1}{\rho}G(z) \\ \frac{1}{\rho}G^T(z) & I \end{bmatrix} > 0 \quad (24)$$

and thus we can take ρ to be z-dependent, e.g.,

$$\rho = \rho(z) > \bar{\sigma}(G(z)). \quad (25)$$

Finding a different ρ for each z is a laborious task. A single constant ρ that will satisfy (11) for all z can be easily be computed by via the matrix inequality

$$\bar{\sigma}(Q) \leq \bar{\sigma}(\text{abs}(Q(z))) \leq \bar{\sigma}(\bar{Q}) \quad (26)$$

where ij-th entry of the the $n \times m$ matrix $\bar{Q}$ is given by

$$[\bar{Q}]_{ij} = \bar{\sigma}(H_{ij}). \quad (27)$$

In view of Lemma 1, the following statement is obvious: Consider the optimization problem

$$\max_{x,y} \ J \triangleq \|x\|^2 + \|y\|^2 \quad (28)$$

subject to

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} I & \frac{1}{\rho}G(z) \\ \frac{1}{\rho}G^T(z) & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 1 \quad \forall z \in \mathcal{R}^k, \ \|z\| = 1. \quad (29)$$

Then, there exists a pair (x, y) solving (1) if, and only if, the global optimum of (28) is strictly greater than 1; note that since all points of the form $(x, y) = (\hat{x}, 0)$ or $(x, y) = (0, \hat{y})$ with $\|\hat{x}\| = \|\hat{y}\| = 1$ satisfy (29), the optimum of (28) is always greater than or equal to 1. Actually, instead of solving the problem (28) for its global optimum, all we need is a point (x, y) where $J(x, y) > 1$. Note that (28) is a nonlinear programming problem where a convex function is to be maximized subject to an infinite number of quadratic constraints (parameterized by z) all of which are ellipsoids centered at the origin.

Problem 1 can also be written as:

Problem 1′ Find an $n \times m$ matrix N of rank 1 (viz., $N = xy^T$) such that for all z with $\|z\| = 1$

$$\begin{aligned} < G(z), N > &= \text{trace}(N^T G(z)) \quad (30) \\ &= x^T G(z) y < 0. \quad (31) \end{aligned}$$

∎

That is, in the Hilbert space of $n \times m$ real matrices with inner product $< B, A >= \text{trace}(A^T B)$, find a hyperplane that strictly separates the origin from the set $\mathbf{W} = \{G(z) \mid \|z\| = 1\}$ and in addition it should hold that the matrix N that defines the perpendicular to this hyperplane is of rank one. In the absence of the restriction $\text{rank}(N) = 1$, the problem may be interpreted as a linear matrix inequality (LMI), which is a special kind of linear programming problem having an infinite number of constraints, viz.

$$\epsilon_* = \min_{\epsilon, N} \epsilon \quad (32)$$

subject to

$$\begin{aligned} < N, G(z) > \leq \ \epsilon, & \quad \forall z, \ \|z\| = 1 \quad (33) \\ |[N]_{ij}| \leq 1, & \quad \forall i = 1, \ldots, n; j = 1, \ldots, m \quad (34) \end{aligned}$$

and a solution exists if and only if the minimal cost $\epsilon_* < 0$. If the rank of N is restricted to be less than $\min\{m, n\}$, the LMI/linear-programming formulation fails. Actually, if the restriction on N is $\text{rank}(N) = \ell < \min\{m, n\}$, then we can write

$$N = \sum_{i=1}^{\ell} x_i y_i^T \quad (35)$$

and thusly transform the problem into one of the same type as (28) or the one described in Lemma 1.

4. A CONCEPTUAL ALGORITHM

In order to solve (1) we can solve a sequence of problems of the type (28), each one having a finite number of inequalities. At the beginning of Step k, assume that the points $z^{(1)}, \ldots, z^{(k-1)}$ have been generated from an arbitrary initial guess $z^{(1)}$ with $\|z^{(1)}\| = 1$. Then, the k-th step of the algorithm is:

Step k Solve for a globally maximizing pair (x, y)

$$J_*^{(k)} = \max J \triangleq \|x\|^2 + \|y\|^2 \tag{36}$$

subject to

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} I & \frac{1}{\rho^{(i)}} G(z^{(i)}) \\ \frac{1}{\rho^{(i)}} G^T(z^{(i)}) & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq 1,$$

$$\forall i = 1, \ldots, (k-1). \tag{37}$$

If the global minimum $J_*^{(k)} = 1$, stop; the problem (28) is infeasible. If $J_*^{(k)} > 1$, let the solution be $(x^{(k)}, y^{(k)})$ and solve

$$\max_{\|z\|=1} z^T \bar{H}(x^{(k)}, y^{(k)}) z \tag{38}$$

where

$$\bar{H}(x^{(k)}, y^{(k)}) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_i^{(k)} y_j^{(k)} H_{ij}. \tag{39}$$

Note that the maximal value in (38) is the maximal eigenvalue $\lambda_{max}(\bar{H})$. Take $z^{(k)} \in \mathcal{R}^k$ to be a maximizing z in (38), viz. $z^{(k)}$ is any unit norm eigenvector of the matrix (39) associated with $\lambda_{max}(\bar{H})$. If $\lambda_{max}(\bar{H}) < 0$, we stop and $x^{(k)}, y^{(k)}$ provide a solution of (1); otherwise, we choose $\rho^{(k)} > \lambda_{max}(\bar{H})$ and go to Step $k+1$. ■

It can be shown that this process will stop in a finite number of steps if (1) has a solution; otherwise, (1) is infeasible.

It should be pointed out that solving (36) for the global maximum may be quite a time consuming problem, although there exist several algorithm for solving nonconvex maximization problems of this type — see, for example, (Horst and Tuy, 1990; Horst *et al.*, 1991; Liu and Papavassilopoulos, 1994).

Notice finally that it is not necessary to solve (36) for the global maximum but we can stop as long as a point $x^{(k)}, y^{(k)}$ with $\|x^{(k)}\|^2 + \|y^{(k)}\|^2 > 1$ has been generated. This may be detrimental to the speed of convergence of the algorithm but avoids spending a lot of time in finding the global maximum of (36). If one chooses to do this, it might be advisable now and then to solve (36) globally.

5. CONCLUSION

The solution of Bilinear Matrix Inequality (BMI) problems has been shown to be equivalent to determining whether the diameter of certain convex set determined by intersection of ellipsoids, centered at the origin in $\mathcal{R}^{n+m}$, is greater than two. This provides a new and conceptually simple geometric perspective on the very broad class of robust control problems that can be formulated as BMI's. Additionally, we have described a conceptual algorithm for determining the radius (half the diameter) of this intersection of ellipsoids. The nonconvex nature of the optimization (36) in our algorithm underscores the fact that, while BMI robust control problems are conceptually simplified by our new geometric interpretation in terms of the diameter of intersections of ellipsoids, the efficient computation of globally optimal solutions to general BMI robust control synthesis problems may still be difficult.

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