Fitting Controllers to Data

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Abstract

The problem of optimally fitting controllers to data is formulated and solved. This formulation is applied to the case in which the class of controllers considered is the same as the one used in model reference adaptive control problem and in which we minimize the induced norm of the control error. The solution to this problem, besides illustrating an use of the saying "let the data speak" to its extremes, gives a robustness-oriented perspective on model reference adaptive control.

Key words: Robust control; identification; uncertain dynamic systems; fitting; learning control; behavioral approach; model reference; adaptive control.

1 Introduction

Fitting models to data is one of the most basic scientific activities, as can be recalled from our first experiments in physics laboratories. This basic activity, in several instances, takes a two sided approach in which models are said to fit or not to fit, as in Popper's philosophy of science, model validation (unfalsification) .... However, in several other instances, we are interested in the questions such as how the model fits a performance criterion and which model fits a criterion in an optimal way."

An interesting development in recent years was the introduction of the unfalsified control paradigm ([1],[2],[3],[4],[5],[6]) which advanced from the model validation (unfalsification) formulation to the controller validation (unfalsification) formulation. In the unfalsified control paradigm, a decision is made about the falsification (unfalsification) of a controller based on the data, the control law and the performance specification. This decision problem is analogous to the two sided fitting models to data problem.

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Since the decision is made upon controllers instead of upon models, we consider it a problem of fitting controllers to data.

Moreover, in the same way as there are instances of the activity of fitting models to data in which we are interested in the model that fits the data in an optimal way, there are instances of the activity of fitting controllers to data in which we are interested in the controller that fits the data in an optimal way.

In this paper, we formulate the problem of optimally fitting controllers to data. This formulation is embedded in the behavioral framework of Willems ([7],[8]) expanded with projection operators to accommodate partial information about signals as in [1]. After introducing the general formulation, we apply it to the case in which the class of controllers considered is the same as the one used in model reference adaptive control problem and in which we minimize an induced norm of the control error.

Let us note that the parameter estimation perspective of continuous-time model reference adaptive control given in [9], which shows "how least squares, as opposed to gradient, estimation can be used in continuous-time adaptive control", can be interpreted as a controller fitting problem in which the norm of the control error is minimized. For this problem, the equations that determine the control law constitute a parameter relation between plant behavior and reference signal. So, given plant behavioral data and a control law, the reference input signal is then *implicitly* determined. In particular, this means any reference signal thusly determined will in general depend on hypothesized control law parameters. That is, a different reference signal is associated, in general, with each candidate controller. Additionally, let us notice that the least squares solution is recovered in our method if there is a controller which leads to a null control error, since in this case the norm and the induced norm of the control error are equal to zero. Moreover, the existence of such a controller is a standard assumption in developments of model reference adaptive control [10].

2 Basic Concepts

2.1 Mathematical Models

The fundamental piece of the "behavioral" framework of Willems ([7],[8]) is the definition of a mathematical model. This definition is formulated according to the black box point of view, "in which we focus on how a system behaves, on the way it interacts with its environment, instead of trying to understand, in the tradition of physics, how it is put together and how its components work" ([7],[8]) . This definition of a mathematical model formalizes, in its ultimate generality, the black box point of view. This ultimate generality can be more strongly evidenced by the fact that Willems backs off "from the usual input/output setting, from the processor point of view, in which systems are seen as influenced by inputs, acting as causes, and producing outputs through these inputs, the internal conditions, and the system dynamics."

Willems starts by assuming that there is a phenomenon to be modeled. Then he "casts the situation in the language of mathematics by assuming that the phenomenon produces elements in a set $\mathbf{U}$" ([7],[8]), called the universum. The elements of $\mathbf{U}$ are called the outcomes of the phenomenon. "A (deterministic) mathematical model for the phenomenon (viewed purely from the behavioral, the black box point of view) claims that certain outcomes are possible, while others are not. Hence a model recognizes a certain subset $\mathcal{B}$ of $\mathbf{U}$. This subset will be called the behavior (of the model)." And the formal definition is given as follows.

Definition 2.1 *A mathematical model is a pair* $(\mathbf{U}, \mathcal{B})$, *with* $\mathbf{U}$ *the universum — its elements are called outcomes — and* $\mathcal{B} \subseteq \mathbf{U}$ *the behavior.*

Definition 2.2 *A controller is a mathematical model.*

With respect to data and measurements, we cite the following excerpts from [7]: "We will now cast measurements in this setting. We will assume that we make certain measurements which we will call the data." "... we ... assume that the data consists of observed realizations of the phenomenon itself. Thus, a data set will be a nonempty subset $\mathcal{D}$ of $\mathbf{U}$. " The formal definition of a data set is, then, given as follows.

Definition 2.3 *A data set is a nonempty subset* $\mathcal{D}$ *of* $\mathbf{U}$.

As in [1], we will work with data information that can evolve with time. Thus we will have a universum of time signals and a data set contained in a time varying projection of this universum.

Definition 2.4 *Given a vector space of time signals* $\mathbf{U}$, *a model* $(\mathbf{U}, \mathcal{B})$, *a mapping* $\mathbf{P}_\tau : \mathbf{U} \to \mathbf{U}$ *and a data set* $\mathcal{D}_\tau \subset \mathbf{P}_\tau(\mathbf{U})$, *we say that the model* $(\mathbf{U}, \mathcal{B})$ *is* unfalsified *by the data set* $\mathcal{D}_\tau$ *if*

$$\mathcal{D}_\tau \subset \mathbf{P}_\tau(\mathcal{B}).$$

Typically $\mathbf{P}_\tau(x)$ is the experimental observation time sampling operator, which returns values of $x(t)$ only for past time intervals over which experimental observations of $x(t)$ have been recorded.

Definition 2.5 *Given a vector space of time signals* $\mathbf{U}$, *a controller* $(\mathbf{U}, \mathcal{B}_c)$, *a desired closed loop behavior* $(\mathbf{U}, \mathcal{B}_d)$, *a mapping* $\mathbf{P}_\tau : \mathbf{U} \to \mathbf{U}$, *and a data set* $\mathcal{D}_\tau \subset \mathbf{P}_\tau(\mathbf{U})$, *we say that a controller* $(\mathbf{U}, \mathcal{B}_c)$ *is* unfalsified *by the data set* $\mathcal{D}_\tau$ *if*

$$\mathbf{P}_\tau((\mathbf{P}_\tau^{-1}(\mathcal{D}_\tau)) \cap \mathcal{B}_c) \subset \mathbf{P}_\tau(\mathcal{B}_d).$$

The data set $\mathcal{D}_\tau$ is the set of actual experimental observations of the plant behavior as observed through the time-sampler $\mathbf{P}_\tau$. Thus, $\mathbf{P}_\tau^{-1}(\mathcal{D}_\tau)$ is the set of behaviors that interpolate the observed data. For example, if we have recorded experimental observations of the first component $x_1(t)$ of a vector-valued signal $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in L_2^n[0, \infty)$ during the time interval $t \in [0, 5]$, then $\mathbf{P}_\tau^{-1}(x)$ is the set of signals $\{y \in L_2^n[0, \infty) \mid y_1(t) = x_1(t) \forall t \in [0, 5]\}$. The set $\mathcal{B}_c$ is the set of signals which satisfy the constraints imposed by the controller c, so definition 2.5 says roughly that a controller is defined to be unfalsified if the set of signals x that are consistent with the data and the controller is, at the past observation times, a subset of a given performance target set $\mathbf{P}_\tau(\mathcal{B}_d)$.

A particularly useful projection operator for dealing with past time only information is the time truncation operator $\mathbf{P}_\tau$ defined by

$$[\mathbf{P}_\tau(x)](t) = \begin{cases} x(t), \text{ if } t \le \tau \\ 0, \quad \text{ if } t > \tau. \end{cases} \tag{1}$$

Definition 2.6 *Given a constant $\sigma > 0$, we define the exponentially-weighted truncated L_2 inner-products $< x, y >_\tau$ and norm $\|x\|_\tau$ by*

$$< x, y >_\tau \triangleq \int_0^\tau e^{-2\sigma(\tau - t)} y^T(t) x(t) \, dt \tag{2}$$

$$\|x\|_\tau \triangleq \sqrt{< x, x >_\tau}. \tag{3}$$

2.2 Best-Fit Controller

As stated in ([7],[8]), the intersection of behaviors is "a way of formalizing that additional laws are imposed on a system." Thus, the role of a controller is to impose constraints on the plant behavior. On the other hand, our goal is to select, based on the data, the constraints imposed by the control law and the performance criterion, the best among the set of given controllers. In order to do that we introduce a cost function and an optimality criterion.

Problem 2.1 *Given a class of controllers $\{(\mathbf{U}, \mathcal{B}_c(\theta)) \mid \theta \in \Theta\}$, where Θ is a set of parameter vectors, the performance (cost) index $\mathcal{I}_\tau$, the operator $\mathcal{E}$, the time truncation operator P_τ, $\tau \in \mathbb{R}_+$, and a data set $\mathcal{D}_\tau \subset P_\tau \mathbf{U}$, find the set of parameters Θ^* such that*

$$\Theta^*(\tau) = \arg\min_{\theta \in \Theta} \mathcal{J}_\tau(\theta) \tag{4}$$

where

$$\mathcal{J}_\tau(\theta) \triangleq \mathcal{E}(\{\mathcal{I}_\tau(b) \mid b \in \mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))\}).$$

and $\mathcal{E}$ denotes either the mean, the max, or the expectation operator and $\mathcal{I}_\tau$ is a functional

$$\mathcal{I}_\tau : \mathbf{P}_\tau(\mathbf{U}) \to \mathbb{R}.$$

When $\mathcal{E}(\cdot)$ is the max operator, then Problem 2.1 is equivalent to an unfalsified control problem with τ-dependent performance criterion $\mathcal{I}_\tau(b) \le \mathcal{J}_\tau^*(\theta) \triangleq \min_\theta \mathcal{J}_\tau(\theta)$. Notice that this is slightly different from the unfalsified control problem formulation [5,6] where a min-max performance criterion is employed, viz., $\min_\theta \max_\tau \mathcal{J}_\tau(\theta)$.

3 Problem Formulation

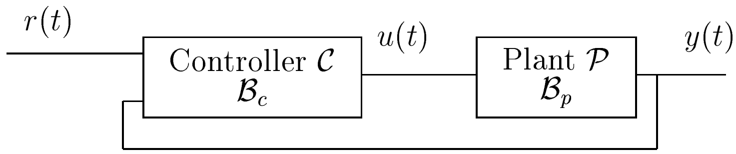

Fig. 1. Feedback control system.

3.1 The System

Consider the system in Figure 1. As in [1], we omit any arrows on the block diagram in Figure 1. This illustrates a departure "from the usual input/output setting, from the processor point of view, in which systems are seen as influenced by inputs, acting as causes, and producing outputs through these inputs, the internal conditions, and the system dynamics ([7],[8])". In particular, in what concerns the problem that we are about to formulate, these remarks indicate that we are interested in relations involving signals instead of functions. More specifically, notice that relations define sets and subsets, which may be used to define the feasible region in an optimization problem.

3.2 The Universum

Let $z = \mathrm{col}(r, y, u) \in \mathbf{U}$ where $\mathbf{U} = \mathcal{R} \times \mathcal{Y} \times \mathcal{U} = \mathcal{L}_{2e}^{n_z}$. Here $\mathcal{R} = \mathcal{L}_{2e}^{n_r}$ is the set of reference signals, $\mathcal{Y} = \mathcal{L}_{2e}^{n_y}$ and $\mathcal{U} = \mathcal{L}_{2e}^{n_u}$ are sets of plant signals, and $n_z = n_r + n_y + n_u$. In this paper we focus on the case $n_r = n_y = n_u = 1$.

3.3 The Data Set

The plant information imposes restrictions only on the past values of the signals u and y. Thus the data set is given by

$$\mathcal{D}_\tau = \{(r, y, u) \in \mathbf{P}_\tau(\mathbf{U}) \mid y = y_{data}, u = u_{data}\},$$

for some

$$(y_{data}, u_{data}) \in \mathbf{P}_\tau(\mathcal{Y} \times \mathcal{U}).$$

3.4 The Performance (Cost) Index

Given a reference model transfer function

$$W_m(s) = k_m Z_m(s)/R_m(s),$$

where, as in [10] and [11], $k_m \in \mathbb{R}\backslash\{0\}$ is a non zero real constant and $R_m(s)$ and $Z_m(s)$ are monic coprime Hurwitz polynomials of degree n and m, respectively.

Let the performance (cost) index

$$\mathcal{I}_\tau((r,y,u)) \triangleq \begin{cases} \|y - w_m * r\|_\tau^2 / \|r\|_\tau^2, & \text{if } \|r\|_\tau \neq 0 \\ 0, & \text{if } \|r\|_\tau = 0 \text{ and } \|y\|_\tau = 0 \\ \infty, & \text{otherwise.} \end{cases}$$

where $w_m = \mathcal{L}^{-1}(W_m(s))$.

3.5 The Class of Candidate Controllers

The class of candidate controllers used is the same as the one used in model reference adaptive control as in [10] and [11]. In order to define the class of candidate controllers, we first define a vector of filters as in [3] and [2]. Notice that we define a vector of filters, and not just a vector of time-domain filtered signals as in [10] and [11]. Let us define $v : \mathcal{L}_{2e} \to \mathcal{L}_{2e}^{n-1}$ by

$$\dot{v}(q) = \Lambda v(q) + lq \tag{5}$$
$$(v(q))(0) = 0 \tag{6}$$

where (Λ, l) is an asymptotically stable system in controllable canonical form, with

$$\lambda(s) = det(sI - \Lambda) = \lambda_1(s) Z_m(s) \tag{7}$$

for some monic Hurwitz polynomial $\lambda_1(s)$ of degree $n^* - 1$, where n^* is the relative degree of $W_m(s)$. Let us also define

$$w(u,y) = (u, v^T(u), y, v^T(y))^T, \quad \text{and} \tag{8}$$
$$\overline{w} = w_m * w. \tag{9}$$

The class of controllers considered is given by $\{(\mathbf{U}, \mathcal{B}_c(\theta)) \mid \theta \in \Theta\}$, where

$$\mathcal{B}_c(\theta) = \{(r,y,u) \mid r = \theta^T w(u,y)\} \quad \text{and}$$

θ is a constant parameter vector in $\mathbb{R}^{2n}$.

Let us notice that the formulation of the class of candidate controllers corresponds exactly to the algebraic part of the model reference adaptive control problem as in [10]. Thus all the signals mentioned in this section may be realized in exactly the same way as they are realized in [10] and [11].

On the other hand, for the analytic part [10], which deals with how the signals evolve with time based in part on assumptions on the plant model, we substitute a controller selection method based solely on data and performance criterion.

4 Problem Solution

4.1 The Set of Optimal Controllers

Theorem 4.1 *The set of parameters* $\Theta^*(\tau)$,

$$\Theta^*(\tau) = \arg\min_{\theta\in\mathbb{R}^{2n}} \mathcal{E}\{\mathcal{I}_\tau(b) \mid b \in P_\tau(P_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))\}$$

is given by

$$\Theta^*(\tau) = \arg\min_{\theta\in\mathbb{R}^{2n}} \{\frac{\theta^T A(\tau)\theta - 2\theta^T B(\tau) + C(\tau)}{\theta^T D(\tau)\theta}\} \tag{10}$$

where

$$A(\tau) = \int_0^\tau e^{-2\sigma(\tau-t)}\overline{w}_{data}\overline{w}_{data}^T dt, \tag{11}$$

$$B(\tau) = \int_0^\tau e^{-2\sigma(\tau-t)}\overline{w}_{data} y_{data} dt, \tag{12}$$

$$C(\tau) = \int_0^\tau e^{-2\sigma(\tau-t)} y_{data}^2 dt, \quad \textit{and} \tag{13}$$

$$D(\tau) = \int_0^\tau e^{-2\sigma(\tau-t)} w_{data} w_{data}^T dt. \tag{14}$$

with

$$\overline{w}_{data} = w_m * w_{data} \quad \textit{and} \tag{15}$$

$$w_{data} = w(u_{data}, y_{data}) \tag{16}$$

provided that $\|u_{data}\|_\tau + \|y_{data}\|_\tau \neq 0$.

Proof. Let us prove, first, that $\mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))$ is a unitary set (i.e., it has one and only one point):

$$\begin{aligned} \mathcal{D}(\tau) &= \{(r,y,u) \in \mathbf{P}_\tau(\mathbf{U}) \mid y = y_{data}, u = u_{data}\} \\ \mathbf{P}_\tau^{-1}(\mathcal{D}_\tau) &= \{(r,y,u) \in \mathbf{U} \mid \mathbf{P}_\tau(y) = y_{data}, \mathbf{P}_\tau(u) = u_{data}\} \\ \mathbf{P}_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta) &= \{(r,y,u) \in \mathbf{U} \mid r = \theta^T w(u,y), \mathbf{P}_\tau(y) = y_{data}, \mathbf{P}_\tau(u) = u_{data}\} \\ \mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta)) &= \{(r,y,u) \in \mathbf{P}_\tau(\mathbf{U}) \mid r = \theta^T w_{data}, y = y_{data}, u = u_{data}\}. \end{aligned}$$

where w_{data} is defined by the equation (16) and by the equations (5) to (6).

Thus $\mathbf{P}_\tau(\mathbf{P}_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))$ is a unitary set, which implies that we can restrict ourselves to the problem of finding the set of parameters $\Theta^*(\tau)$ such that

$$\begin{aligned}\Theta^*(\tau) &= \arg\min_{\theta\in\Theta}\{\mathcal{I}_\tau(b) \mid b \in P_\tau(P_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))\} \\ &= \arg\min_{\theta\in\Theta}\{\frac{\|[y_{data} - w_m * (\theta^T w_{data})]\|_\tau{}^2}{\|(\theta^T w_{data})\|_\tau{}^2}\} \\ &= \arg\min_{\theta\in\Theta}\{\frac{\|(y_{data} - \theta^T \overline{w}_{data})\|_\tau{}^2}{\|(\theta^T w_{data})\|_\tau{}^2}\} \\ &= \arg\min_{\theta\in\Theta}\{\frac{\theta^T A(\tau)\theta - 2\theta^T B(\tau) + C(\tau)}{\theta^T D(\tau)\theta}\},\end{aligned}$$

where $A(\tau)$, $B(\tau)$, $C(\tau)$, $D(\tau)$, $\overline{w}_{data}$, and w_{data} are defined by the equations (11) to (16). □

4.2 Matrices Properties

Property 4.1 *The matrices $D(\tau)$ and*

$$\begin{bmatrix} A(\tau) & -B(\tau) \\ -B^T(\tau) & C(\tau) \end{bmatrix}$$

are symmetric and positive semidefinite.

Proof. A simple inspection reveals that these matrices are symmetric matrices. The positive semidefiniteness of these matrices follows by observing that

$$\begin{bmatrix} \theta \\ 1 \end{bmatrix}^T \begin{bmatrix} A(\tau) & -B(\tau) \\ -B^T(\tau) & C(\tau) \end{bmatrix} \begin{bmatrix} \theta \\ 1 \end{bmatrix} = \|[y_{data} - w_m * (\theta^T w_{data})]\|_\tau{}^2 \geq 0 \tag{17}$$

$$\theta^T D(\tau)\theta = \|\theta^T w_{data}\|_\tau{}^2 \geq 0 \tag{18}$$

□

Property 4.2 *The null space of $D(\tau)$ is contained in the null space of $A(\tau)$.*

Proof.

$$\begin{aligned} D(\tau)\theta = 0 & \\ &\Rightarrow \theta^T D(\tau)\theta = 0 \\ &\Rightarrow \|(\theta^T w_{data})\|_\tau{}^2 = 0 \\ &\Rightarrow \|[w_m * (\theta^T w_{data})]\|_\tau{}^2 = 0 \\ &\Rightarrow \|(\theta^T \overline{w}_{data})\|_\tau{}^2 = 0 \\ &\Rightarrow \theta^T A(\tau)\theta = 0 \\ &\Rightarrow A(\tau)\theta = 0, \end{aligned}$$

since $A(\tau)$ is symmetric and positive semidefinite by property 4.1. □

Property 4.3 *The null space of $A(\tau)$ is contained in the null space of $B^T(\tau)$.*

Proof.

$$
\begin{aligned}
A(\tau)\theta = 0 & \\
& \Rightarrow \theta^T A(\tau)\theta = 0 \\
& \Rightarrow \|\theta^T \overline{w}_{data}\|_{\tau}^{2} = 0 \\
& \Rightarrow < y, (\theta^T \overline{w}_{data}) >_{\tau} = 0 \\
& \Rightarrow B^T(\tau)\theta = 0.
\end{aligned}
$$

□

Definition 4.1 *Let us define $\lambda_{\min}(\cdot,\cdot)$ as the operator that returns the least generalized eigenvalue of two symmetric semipositive definite matrices, $\theta_{\min}(\cdot,\cdot)$ as the operator that returns one of the eigenvectors associated with that eigenvalue, and $\theta^1_{\min}(\cdot,\cdot)$ as the operator that either returns the eigenvector associated with that eigenvalue which has its last component equal to one if this exists or returns a null vector otherwise.*

Theorem 4.2 *The optimal cost is*

$$
\lambda_{opt} \triangleq \inf_{\theta \in \mathbb{R}^{2n}} \mathcal{E}\{\mathcal{I}_\tau(b) \mid b \in P_\tau(P_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))\} = \lambda_{\min}(\overline{A}_{pen}(\tau), \overline{E}_{pen}(\tau))
$$

where

$$
\overline{A}_{pen}(\tau) = \begin{bmatrix} \overline{A}(\tau) & -\overline{B}(\tau) \\ -\overline{B}^T(\tau) & \overline{C}(\tau) \end{bmatrix} \tag{19}
$$

$$
\overline{E}_{pen}(\tau) = \begin{bmatrix} \overline{D} & 0 \\ 0 & 0 \end{bmatrix} \tag{20}
$$

and

$$
\overline{A}(\tau) \triangleq \mathcal{D}_{||}^T(\tau) A(\tau) D_{||}(\tau) \tag{21}
$$

$$
\overline{D}(\tau)(\tau) \triangleq \mathcal{D}_{||}^T D(\tau) D_{||}(\tau) \tag{22}
$$

$$
\overline{B}(\tau) \triangleq \mathcal{D}_{||}^T(\tau) B(\tau) \tag{23}
$$

and $D_{||}(\tau)$ denotes the orthogonal projection onto the range space of $D(\tau)$.[3] *Moreover,for $C(\tau) > 0$, when a cost-minimizing parameter vector θ exists, it satisfies*

$$
\theta_{opt} = D_{||}(\tau)\overline{\theta}_{opt} \tag{24}
$$

where

$$
\begin{bmatrix} \overline{\theta}_{opt} \\ 1 \end{bmatrix} = \theta^1_{\min}(\overline{A}_{pen}(\tau), \overline{E}_{pen}(\tau))
$$

[3] The MATLAB function `orth(D)` computes the othogonal projection of a matrix D onto its range.

On the other hand, for $C(\tau) = 0$, we have that cost-minimizing parameter vectors θ exist and are given by

$$\theta_{opt} = D_{\|}(\tau)\overline{\theta}_{opt} \tag{25}$$

where

$$\overline{\theta}_{opt} = \theta_{\min}(\overline{A}(\tau), \overline{D}(\tau)).$$

Proof. For $C(\tau) > 0$, and $\alpha \geq 0$, let

$$J(\theta, \alpha) \triangleq \{\theta^T A(\tau)\theta - 2\theta^T B(\tau) + C(\tau) - \alpha\theta^T D(\tau)\theta\}. \tag{26}$$

Then, the cost $\mathcal{E}\{\mathcal{I}_\tau(b) \mid b \in P_\tau(P_\tau^{-1}(\mathcal{D}_\tau) \cap \mathcal{B}_c(\theta))\}$ is less than or equal to α if, and only if, $J(\theta, \alpha) \leq 0$. From Properties 4.2 and 4.3, we have

$$J(\theta, \alpha) = J(\overline{\theta}, \alpha) \triangleq \{\overline{\theta}^T \overline{A}(\tau)\overline{\theta} - 2\overline{\theta}^T \overline{B}(\tau) + C(\tau) - \alpha\overline{\theta}^T \overline{D}(\tau)\overline{\theta}\}. \tag{27}$$

where $\overline{\theta} \triangleq D_{\|}^T\theta$. Using property 4.1, it follows that the the infimal cost is $\lambda_{\text{opt}} = \alpha = \lambda_{\min}(\overline{A}_{\text{pen}}(\tau), \overline{E}_{\text{pen}}(\tau))$ and the corresponding cost-minimizing θ, when it exists, is $D_{\|}(\tau)\overline{\theta}_{\text{opt}}(\tau)$.

On the other hand, for $C(\tau) = 0$, which implies that $B(\tau) = 0$, we have that

$$\overline{\theta}_{\text{opt}} = \theta_{\min}(\overline{A}(\tau), \overline{D}(\tau)).$$

□

5 Practical Considerations

Let us comment on some practical aspects of the theory. To begin with, let us recall that we have not introduced any plant model, so that state and noise are terms that are not used in our formulation. However, in order to answer some practical questions, let us assume that we are dealing with a plant having an unknown state space model with an uncertain initial state plus noise.

It is important that the "signal to noise" ratio be sufficiently great to ensure that noise and/or an initial state uncertainty does not dominate. More precisely, the criterion $\mathcal{I}_\tau((r, y, u))$ cannot be expected to produce a useful and appropriate ordering of control laws when the energy in the plant input signal $\|u_{data}\|_\tau^2$ is too small to dominate over the effects of plant noise and plant initial state on the observed plant output energy $\|y_{data}\|_\tau^2$. Additionally, the input signal u_{data} also should have a sufficiently rich spectral content to be capable of exciting closed-loop system modes which might contribute a large value of $\mathcal{I}_\tau((r, y, u))$. This latter requirement is analogous to the "persistent excitation" condition that arises elsewhere in the theory of system identification and adaptive control.

The exponential weighting parameter σ in (2)– (3) is used in our theory to ensure that we keep learning from new data, and that the influence of very old data is appropriately small. Typically, σ would be chosen to be roughly the reciprocal of the desired control bandwidth, but it can also be used to fine tune the tradeoff between performance and noise sensitivity. A smaller value for sigma tends to increases sensitivity to high-frequency noise, and a larger value tends to decrease it.

Finally, it should be noted that our controller fitting results can only be expected to be good in achieving the goal of making $\mathcal{I}_\tau((r,y,u))$ small when the class $\{(\mathbf{U},\mathcal{B}_c(\theta)) \mid \theta \in \Theta\}$ of candidate controllers is sufficiently rich that it includes at least one controller $\mathcal{B}_c(\theta)$ that is capable of achieving this goal. In many applications, the class of linear time-invariant controllers of order two or three may suffice. When it does not, then one must augment or modify the class of candidate controllers, e.g., by including higher order controllers.

6 Example

Let us choose $W_m(s) = \frac{1}{s+1}$, $\Lambda = -1$, $l = 1$ and $\sigma = 0.01$. The filter w defined in section 3.5 is then given by $w = (u, y)$ and the class of candidate controllers is given by $\{(\mathbf{U},\mathcal{B}_c(\theta)) \mid \theta \in \Theta\}$, where $\mathcal{B}_c(\theta) = \{(r,y,u) \mid r = \theta^T(u,y)\}$ and θ is a constant parameter vector in $\mathbb{R}^2$. For purposes of this simulation let "the true but unknown plant" be given by

$$\dot{x} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x$$

where n_1 and n_2 are uncorrelated normally distributed random signals with mean zero and variance one. Let the reference signal $r(t) = signal(cos(0.1\pi t))\, \forall t \geq 0$. We obtain (y_{data}, u_{data}) by closing the loop with the initial controller associated to the parameter $\theta(0) = \begin{pmatrix} 1 & 0 & 0.1 & -0.1 \end{pmatrix}^T$ and the initial plant state given by $x = 0$. Thus we are able to use our theory to compute a new controller parameter $\theta(t)$ based on the data available at any given time $t = \tau$. Controller adaptation is achieved by repeating this operation periodically as time τ evolves and (y_{data}, u_{data}) accumulates, in order to update the controller parameter θ. Using this procedure to update the controller parameter vector $\theta(t)$ every 5 sec starting at time $\tau = 5$, we obtained the simulation results shown in figure 2.

7 Concluding Remarks

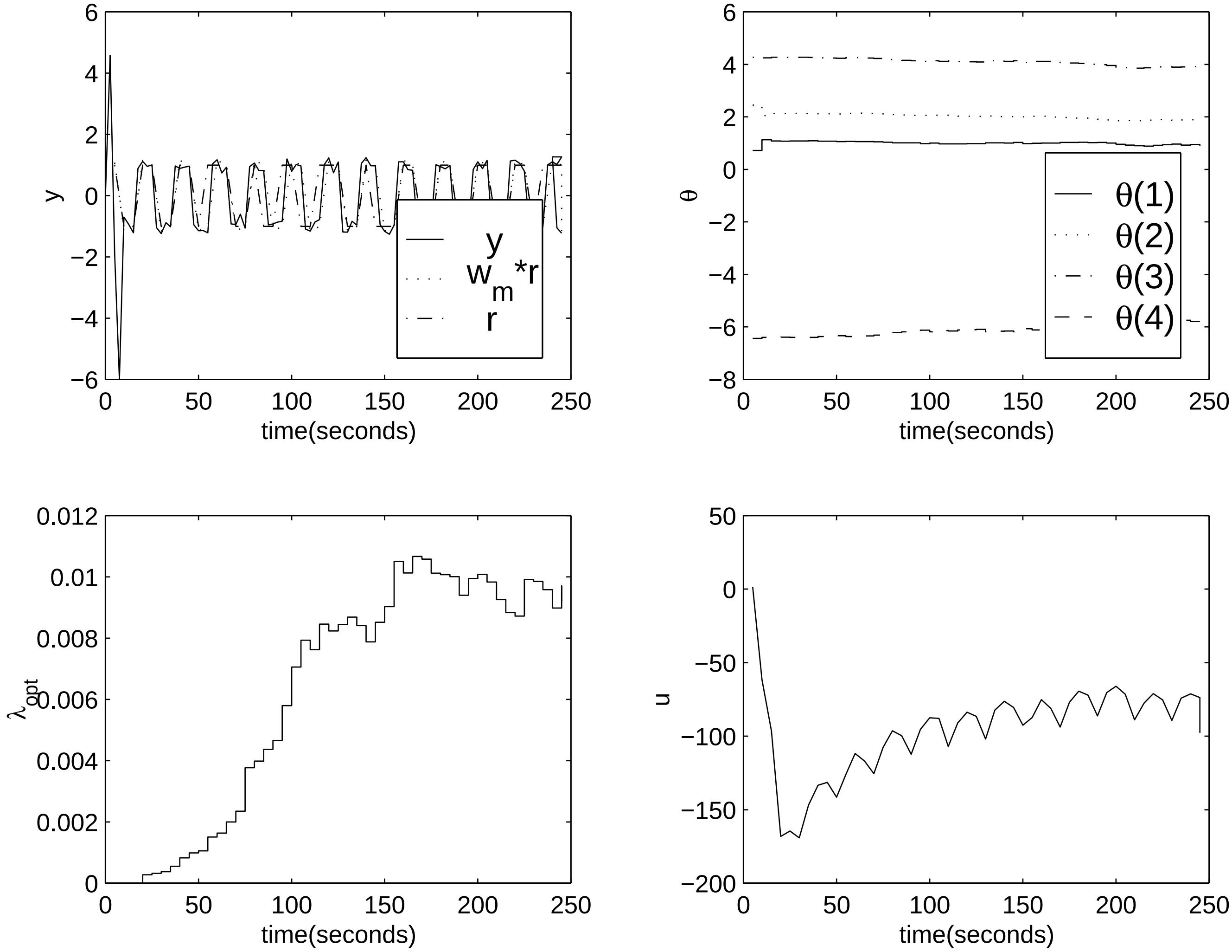

Fig. 2. Simulation Results

In this work, we formulated a problem of optimally fitting controllers to data. We used open-loop plant input-output data, collected during closed-loop operation. The quality of the fit was determined by a quadratic closed-loop performance index, with exponential de-weighting of old data. The saying "let the data speak" ([7],[8]) was used to its extremes. We focused on the application of this general formulation to the case in which the class of candidate controllers is similar to the one used in the model reference adaptive control problem, except that we seek to minimize the induced norm of command to the control error transfer function, instead of the norm of the control error signal itself. No assumptions were made on the plant. We proceeded to a complete analysis of the resultant optimization problem. As expected, the least squares solution was obtained as a particular solution. Moreover, by interpreting the parameter estimation perspective of continuous-time model reference adaptive control given in [9] as a controller fitting problem, we gave an additional perspective on continuous-time model reference adaptive control, a perspective particularly related to questions of robustness.

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