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Zames–Falb multipliers for MIMO nonlinearities‡

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SUMMARY

In their celebrated 1968 paper on nonlinear stability, Zames and Falb determined a class of multipliers that preserve positivity of monotone SISO nonlinearities. They conjectured that their results might also hold for incrementally positive, norm-bounded MIMO nonlinearities. In this note, we demonstrate that their conjecture regarding MIMO nonlinearities holds true only if a further restriction is applied. Specifically, we show that it suffices either to restrict the nonlinearity to be the gradient of a convex real-valued function or to restrict the multiplier to be a real-valued function of frequency. Copyright © 2000 John Wiley & Sons, Ltd.

KEYWORDS: nonlinear systems; stability; robustness; multipliers; integral quadratic constraints

1. INTRODUCTION

A class of non-causal multipliers was introduced by Zames and Falb [1] to study stability of the system shown in Figure 1 having a stable, linear-time-invariant (LTI) plant H in the feed-forward path and a memoryless, monotone nonlinearity N in the feedback path (see Figure 1). Briefly speaking, the Zames–Falb multiplier approach to determining stability of a system rests on finding a class $\mathscr{M}$ of possibly non-causal, linear-time-invariant multipliers that is *positivity preserving* for N in the sense that $M \in \mathscr{M}$ implies positivity of the operator M^*N. Additionally, the multipliers $M \in \mathscr{M}$ are required to be factorizable as

$$M = M_- M_+ \tag{1}$$

where M_-, M_+ have the properties that

$$M_-, M_+ \text{ are invertible} \tag{2}$$

$$M_+, M_+^{-1}, M_-^*, M_-^{*-1} \text{ are causal and have finite gain.} \tag{3}$$

These properties ensure that for any such multiplier, stability of the system shown in Figure 1 is equivalent to that of the system shown in Figure 2. Stability of the system then follows if (see

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‡ Dedicated to the memory of George Zames

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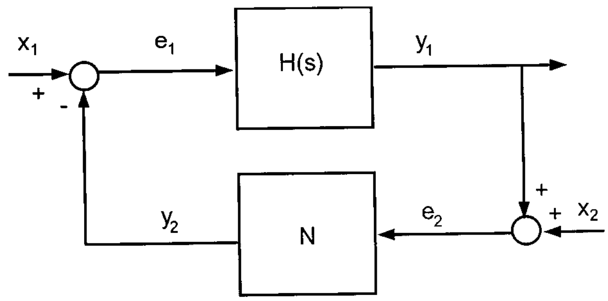

Figure 1. The feedback interconnection has a stable, LTI plant $H(s)$ and a memoryless nonlinearity N.

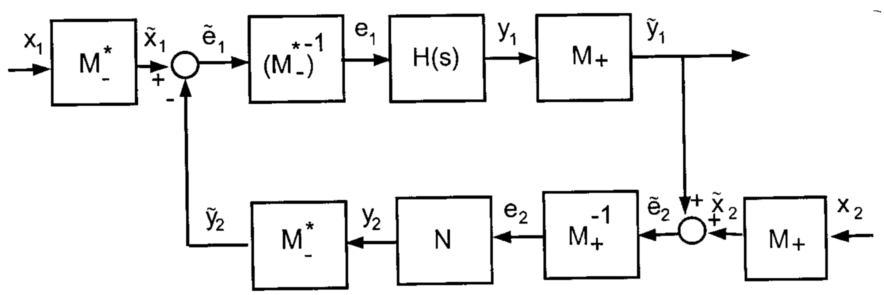

Figure 2. Equivalent transformed feedback interconnection with multipliers. The multipliers M_+, M_+^{-1}, M_-^*, M_-^{*-1} are causal and stable with finite gain.

Theorem 2 of [1]) MH is strongly positive and N has finite gain. In Reference [1], the results were proved for single-input single-output (SISO) nonlinearity N and it was claimed, as a concluding remark, that in the multivariable case

... the mapping N is no longer defined by a scalar function N but by a vector function N_v with the properties:

$$N_v(0) = 0 \tag{4}$$

$$\langle r - s, N_v(r) - N_v(s) \rangle \geqslant 0 \tag{5}$$

there is some constant $c \geqslant 0$ such that

$$\| N_v(r) \| \leqslant c \| r \| \quad \textit{for all } r \tag{6}$$

A careful perusal of our proofs will clearly indicate the validity of this generalization [1].

In fact, this claim is incorrect without further restrictions on N_v, as we shall show in Section 2.

Historically, stability analysis of feedback systems having feedforward element whose dynamics are represented by a differential equation and feedback element which is a scalar nonlinearity with its map restricted to the first and third quadrants was first considered by Lur'e and has been a well-studied topic since then (see References [1–4] and references therein). A well-known graphical criterion for such systems is the Popov criterion [5, Chapter 6, p. 186]. Zames and Falb [1] considered the same set-up except for the fact that the class of nonlinearities was replaced by a subclass, defined by (4)–(6). They used a then novel operator approach to examine the absolute stability of the system and explicitly proved the results for the *scalar* case. For this class of feedback systems, their results are the sharpest available in SISO setting and the constraints imposed by the circle criterion, the Popov criterion and the off-axis circle criterion are obtained as an instance (or as the limiting case, for the latter two criteria) of the ones imposed by the Zames–Falb multipliers [6]. In addition, algorithms for a practical usage of these multipliers exist (see Reference [4] and references therein). The problem of extending the Popov criterion to the multivariable case has received considerable attention (see Reference [7] and references therein) but seemingly no effort has gone into extending the Zames–Falb multipliers likewise although use of diagonal, frequency dependent scaling matrices, closely related to the Zames–Falb multipliers, for such nonlinearities has been well known (see Reference [8] and references therein). Recently, extensions to the special case of systems which have multiple *scalar* nonlinearities, every one of which is described by Equations (4)–(6) modulo the loop-shifting transformation, have been investigated. As a case in point, Haddad and Kapila [9] have considered the class of the nonlinear feedback operators defined to be

$$\Phi \doteq \{\phi : \mathbb{R}^n \to \mathbb{R}^n | \phi(y) = \mathrm{diag}\,(\phi_i(y_i)), \quad \phi_i(\cdot) \in \mathscr{C}^1, 0 < \phi_i'(y_i) < \mu_i < \infty \quad \forall y_i \in \mathbb{R}\},$$

where $y \doteq [y_1\, y_2, \ldots, y_n]^{\mathrm{T}}$ and μ_i are the upper bounds on the slope of the scalar nonlinearities ϕ_i, and have derived sufficiency conditions for the absolute stability of the feedback system using a so-called kinetic Lyapunov function. Apparently motivated by the analysis of a simple Hopfield network and Chua's circuit, Suykens *et al.* [10] have investigated exactly the same problem and have used a Yakubovic type Lyapunov function to derive sufficiency conditions for the absolute stability of the feedback system. D'Amato *et al.* [11] have focused on a subset of the above class of feedback nonlinearities, viz. the class of *repeated* scalar nonlinearities which is obtained by setting $\phi_i(\cdot) = \phi_j(\cdot)\,\forall i, j$. They follow the Zames–Falb approach to derive the integral quadratic constraints (IQCs) satisfied by such diagonal operators and conclude the absolute stability of the feedback system as an application of Theorem 1 of Reference [12].

However, literature on multipliers for the general case i.e. non-diagonal MIMO nonlinearities is sparse. While re-examining positivity of the scalar operators relevant to the system shown in Figure 1 in SISO setting, Willems [13] acknowledged that difficulties might exist in extending some of the results, e.g. Lemma 8 of [1], to MIMO case (see Reference [13, Chapter 3, p. 66]) and the question whether the Zames–Falb claim for the MIMO case is true or false has remained unsettled thus far. In this note, we show that such a multiplier *can* fail to preserve positivity of the MIMO nonlinearity. However, if the MIMO nonlinearity is additionally stipulated to be the gradient of a potential function, it can be shown that the Zames–Falb multipliers do indeed preserve its positivity.

The paper is organized as follows. We begin with a simple counterexample to the Zames–Falb conjecture in Section 2. Notation and a preliminary lemma are introduced in Section 3. The problem formulation is given in Section 4. Main results are derived in Section 5 and briefly discussed in Section 6. Conclusions are in Section 7. Relevant background results may be found in the Appendix.

2. MIMO COUNTEREXAMPLE

A counterexample to the Zames–Falb claim for the MIMO nonlinearities satisfying (4)–(6) is provided by the trivial *linear* nonlinearity $N_v:\mathbb{R}^2 \to \mathbb{R}^2$

$$N_v(x_1, x_2) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

Nonlinearity (7) satisfies (4)–(6), yet the operator $M^* N_v$ fails to be positive for any Zames–Falb multiplier having a non-zero imaginary part, $\mathrm{Im}(\hat{m})(\mathrm{j}\omega) \neq 0$. Indeed, the Hermitian part of its frequency-response matrix has a negative eigenvalue

$$\lambda_{\min}\left(\mathrm{herm}\left(\hat{m}^*(\mathrm{j}\omega)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\right) \tag{8}$$

$$= -|\mathrm{Im}(\hat{m}(\mathrm{j}\omega))| \tag{9}$$

$$< 0 \tag{10}$$

whence, by Lemma A1, $M^* N_v$ cannot be a positive operator.

A careful scrutiny of the proofs in [1] reveals the source of the difficulty in extending the Zames–Falb proofs to the MIMO case. The problem arises due to the fact that a preliminary result used to establish positivity of $M^* N$, Lemma 7 of Reference [1], states

$$xN(x) - yN(x) \geqslant P(x) - P(y) \quad \forall x, y \in \mathbb{R} \tag{11}$$

where $P(x) \doteq \int_0^x N(\zeta)\,\mathrm{d}\zeta$. In the MIMO case $x, y \in \mathbb{R}^n$, the integral in (11) is path-independent if (and only if) skew $N_v'(x) \equiv 0$ (see Lemma A4 in Appendix). Therefore, the potential function $P(x)$ *may not even exist* so that the Lemma 7 of Reference [1] argument is invalid for the MIMO case.

3. PRELIMINARIES

Notation used is summarized in Appendix B.

Definition 1

An operator $F: L_2 \to L_2$ is said to be *positive* if $\langle x, Fx \rangle \geqslant 0\, \forall x \in L_2$. If, additionally, there exists a constant $\delta > 0$ such that $\langle x, Fx \rangle \geqslant \delta \| x \|^2\, \forall x \in L_2$, then F is said to be *strongly positive*.

The constraint (5) is called an *incremental positivity* condition [14]. It is a MIMO generalization of the SISO property of monotonicity.

Definition 2

$\mathscr{M}_{\mathrm{odd}}$ denotes the class of transfer functions (convolution operators) $M: x \mapsto m * x$ where $\hat{m}: \mathbb{R} \to \mathbb{R}$ is defined as

$$\hat{m}(\mathrm{j}\omega) \doteq m_0 - \hat{z}(\mathrm{j}\omega)\, \forall \omega \tag{12}$$

and

$$m_0 - \| z \|_1 > 0. \tag{13}$$

The subclass obtained under the restriction $z(t) \geqslant 0\, \forall t$ is designated $\mathscr{M}$. The elements of $\mathscr{M}$ and $\mathscr{M}_{\text{odd}}$ are called *Zames–Falb multipliers.*

Remark 1

When operating on a vector-valued signal $y(t) \in \mathbb{R}^n$, the operation of multiplier $M \in \mathscr{M}_{\text{odd}}$ is component wise on y; equivalently, one may regard the multiplier M as a MIMO operator having the $n \times n$ MIMO frequency response matrix

$$\hat{m}(j\omega) I$$

where $\hat{m}(j\omega)$ is the scalar-valued frequency response described in (12) and (13).

Remark 2

The original definition of $\mathscr{M}$ [1] allows terms of the form $\sum_{i=1}^{\infty} z_i e^{-j\omega\tau_i}$ where τ_i is a sequence in $[0, \infty)$ and $\{z_i\}$ is a sequence in l_1. But, these terms can be included as impulses in $z(\cdot)$ itself leading to an ease of notation. Note that $M \in \mathscr{M}$ implies M satisfies (1)–(3).

Remark 3

It may be observed that the Nyquist locus of a Zames–Falb multiplier lies in the open right-half s-plane inside a disk centred at $m_0 > 0$ having radius $\|z\|_1 < m_0$.

Definition 3

$\mathscr{N}$ denotes the class of MIMO nonlinearities $N_v : \mathbb{R}^n \to \mathbb{R}^n$ for which (4)–(6) hold and $\mathscr{N}_{\text{odd}}$ denotes its subclass

$$\mathscr{N}_{\text{odd}} \doteq \{N_v \in N : N_v(x) = -N_v(-x)\, \forall x \in \mathbb{R}^n\}$$

In the remainder of the paper, we shall denote the MIMO nonlinearity as simply N rather than N_v, for the ease of notation. The following lemma provides a useful MIMO generalization of certain features of Lemmas 7 and 8 of Reference [1].

Lemma 1

Let $P : \mathbb{R}^n \to \mathbb{R}$, $P \in \mathscr{C}^1$ be a convex function, $x \in \mathbb{R}^n$. Define $y \doteq N(x) \doteq (P'(x))^T$. Then

$$\text{trace}\,[r_{yx}(0) - r_{yx}(\tau)] \geqslant 0 \quad \forall \tau \in \mathbb{R}. \tag{14}$$

Furthermore if N is odd, then

$$\text{trace}\,[r_{yx}(0)] \geqslant |\text{trace}\, r_{yx}(\tau)]| \quad \forall \tau \in \mathbb{R}. \tag{15}$$

Proof. P convex implies (see Reference (15)) $P(\tilde{x}) \geqslant P(x) + P'(x)(\tilde{x} - x)$, $\forall x, \tilde{x} \in \mathbb{R}^n$. In particular,

$$(x(t) - x(t+\tau))^{\mathrm{T}}\, y(t) \geqslant P(x(t+\tau)) - P(x(t)). \tag{16}$$

Therefore,

$$\int_{-\infty}^{\infty} (x(t) - x(t+\tau))^{\mathrm{T}}\, y(t)\, dt \geqslant \int_{-\infty}^{\infty} (P(x(t+\tau)) - P(x(t)))\, dt \tag{17}$$

i.e. $\text{trace}\,[r_{yx}(0) - r_{yx}(-\tau)] \geqslant 0$.

If in addition N is odd, then P is an even function so that, in particular, the inequality $(x(t) + x(t + \tau))^{\mathrm{T}} y(t) \geqslant P(x(t + \tau)) - P(x(t))$ holds along with (16) $\forall t, \tau \in \mathbb{R}$; whence,

$$x^{\mathrm{T}}(t)\, y(t) + |x^{\mathrm{T}}(t + \tau)\, y(t)| \geqslant P(x(t + \tau)) - P(x(t)). \tag{18}$$

Integrating (16) and (18) w.r.t. t as in (17), the results (14) and (15) follow. □

4. PROBLEM FORMULATION

The problems of interest are as follows:

Problem 1

Find the greatest subclass $\tilde{\mathcal{N}} \subset \mathcal{N}$ (or $\tilde{\mathcal{N}}_{\mathrm{odd}} \subset \mathcal{N}_{\mathrm{odd}}$) whose positivity is preserved by *every* multiplier $M \in \mathcal{M}$ (or, respectively, $M \in \mathcal{M}_{\mathrm{odd}}$).

Problem 2

Find the greatest subclass $\tilde{\mathcal{M}} \subset \mathcal{M}$ (or $\tilde{\mathcal{M}}_{\mathrm{odd}} \subset \mathcal{M}_{\mathrm{odd}}$) which is positivity preserving for *all* nonlinearities in N (or, resp. $\mathcal{N}_{\mathrm{odd}}$).

Observe that [1] claimed $\tilde{\mathcal{N}} = \mathcal{N}\, (\tilde{\mathcal{N}}_{\mathrm{odd}} = \mathcal{N}_{\mathrm{odd}})$ and, alternatively, $\tilde{\mathcal{M}} = \mathcal{M}\, (\tilde{\mathcal{M}}_{\mathrm{odd}} = \mathcal{M}_{\mathrm{odd}})$.

5. MAIN RESULT

Theorem 1

Suppose $N \in \mathcal{N}$, $N \in \mathscr{C}^1$ (or $N \in \mathcal{N}_{\mathrm{odd}}$, $N \in \mathscr{C}^1$). Then, M^*N is positive for all $M \in \mathcal{M}$ (or, resp. $M \in \mathcal{M}_{\mathrm{odd}}$) if and only if,

$$\operatorname{skew}(N'(x)) = 0 \quad \forall x. \tag{19}$$

Proof. (*if*) Suppose $\operatorname{skew}(N'(x)) = 0\, \forall x$. Define $y = N(x)$. Then,

$$\begin{aligned}
\langle M^*N(x), x\rangle &= \langle M^* y, x\rangle \\
&= \langle m_0^* y, x\rangle - \langle z^* * y, x\rangle \\
&= \operatorname{trace}[m_0 r_{yx}(0)] - \operatorname{trace}\left[\int_{-\infty}^{\infty} z(-t)\, r_{yx}(-t)\, \mathrm{d}t\right] && (20)\\
&= \operatorname{trace}\left[(m_0 - \int_{-\infty}^{\infty} z(t)\, \mathrm{d}t)\, r_{yx}(0)\right] + \operatorname{trace}\left[\int_{-\infty}^{\infty} z(t)\,(r_{yx}(0) - r_{yx}(t))\, \mathrm{d}t\right] \\
&= \left(m_0 - \int_{-\infty}^{\infty} z(t)\, \mathrm{d}t\right) \operatorname{trace}[r_{yx}(0)] + \int_{-\infty}^{\infty} z(t)\, \varepsilon(t)\, \mathrm{d}t \\
&\geqslant (m_0 - \|z\|_1) \operatorname{trace}[r_{yx}(0)] + \int_{-\infty}^{\infty} z(t)\, \varepsilon(t)\, \mathrm{d}t. && (21)
\end{aligned}$$

Since $\text{skew}(N'(x)) = 0\ \forall x$, Lemma A4 (see Appendix A) ensures that there exists a continuously differentiable function $P:\mathbb{R}^n \to \mathbb{R}$ such that N is the gradient of it. Furthermore, condition (5) on N implies that P is convex [15, Chapter 6, p. 116]. Then, Lemma 1 ensures that $\varepsilon(t) \doteq \text{trace}[r_{xy}(0) - r_{xy}(t)] \geqslant 0\ \forall t$, (4) and (5) insure that $\text{trace}[r_{yx}(0)] \geqslant 0$, and (13) ensures that $m_0 - \|z\|_1 > 0$. If $M \in \mathscr{M}$ then $z(t) > 0\ \forall t$ and positivity of M^*N follows from (21).

It remains to establish positivity of M^*N for the case where $M \in \mathscr{M}_{\text{odd}}$. Consider (20). Using (15), it follows that

$$\begin{aligned}\langle M^*N(x), x\rangle &= \langle M^*y, x\rangle \\ &\geqslant \text{trace}\,[m_0 r_{yx}(0)] - \text{trace}\left[\int_{-\infty}^{\infty} |z(-t)| r_{yx}(0)\,\mathrm{d}t\right] \\ &\geqslant \text{trace}\left[\left(m_0 - \int_{-\infty}^{\infty} |z(-t)|\,\mathrm{d}t\right) r_{yx}(0)\right] \\ &= (m_0 - \|z\|_1)\,\text{trace}\,[r_{yx}(0)].\end{aligned}$$

Since $m_0 - \|z\|_1 > 0$ and $\text{trace}\,[r_{yx}(0)] \geqslant 0$, we may conclude that M^*N is positive.

(*Only if*) Suppose $\text{skew}(N'(x)|_{x_0}) \neq 0$ for some $x_0 \in \mathbb{R}^n$. We shall demonstrate that for $x_\tau(t) \doteq P_\tau(x_0 + \varepsilon y(t))$ (where $y(t) \doteq \text{Re}(y_0 \mathrm{e}^{\mathrm{j}\omega_0 t})$), $\langle M^*Nx_\tau, x_\tau\rangle < 0$ for some y_0, all τ sufficiently large, and all ε sufficiently small. The basic idea of the proof is to choose M from a sequence of Zames–Falb multipliers converging pointwise to $\hat{m}(\mathrm{j}\omega) = \mathrm{j}\omega$ so that $M(0) \to 0$, $M(\mathrm{j}\omega_0) \to \mathrm{j}\omega_0$ and

$$\lim_{\tau\to\infty} \tfrac{1}{\tau}\langle M^*Nx_\tau, x_\tau\rangle = 2\varepsilon^2 y_0^*\left(-\mathrm{j}\omega_0\,\text{skew}\left(\frac{\partial N}{\partial x}(x_0)\right)\right) y_0 + \mathrm{O}(\varepsilon^3)$$

from which we conclude, via Lemma A1, that M^*N is not positive, since the Hermitian matrix $(-\mathrm{j}\omega_0\,\text{skew}(N'(x)|_{x_0}))$ is indefinite. The details follow.

Write $N'(x)|_{x_0} = A_\mathrm{h} + A_\mathrm{s}$ where $A_\mathrm{h} \doteq \text{herm}(N'(x)|_{x_0})$ and $A_\mathrm{s} \doteq \text{skew}(N'(x)|_{x_0})$. Observe that $A_\mathrm{s}, A_\mathrm{h} \in R^{n\times n}$ with A_h positive semidefinite and jA_s indefinite. Since $N:\mathbb{R}^n \to \mathbb{R}^n$, A_s has diagonal entries $(A_\mathrm{s})_{ii} = 0\ \forall i$. W.l.o.g. suppose $(A_\mathrm{s})_{12} = -a$ so that $(A_\mathrm{s})_{21} = a$ where $a > 0$. Now, let $x(t) = x_0 + \varepsilon y(t)$. Let $y(t) = \text{Re}\,(y_0 \mathrm{e}^{\mathrm{j}\omega_0 t})$ where

$$y_0 = [1\ \ j\ \ 0 \cdots 0]^\mathrm{T}. \tag{22}$$

Define $x_\tau \doteq P_\tau x, y_\tau \doteq P_\tau y$. Note that $x_\tau, y_\tau \in L_2$. Consider the ε-dependent multiplier M_ε whose Fourier transform is given as

$$\hat{m}_\varepsilon(j\omega) = \frac{j\omega + \varepsilon^3}{\varepsilon^3 j\omega + 1}, \quad 0 < \varepsilon < 1. \tag{23}$$

For this multiplier, $m_0 = \varepsilon^{-3}\delta(t), z(t) = (\varepsilon^{-6} - 1)\mathrm{e}^{-t/\varepsilon^3}u(t)$ where $u(t)$ is the unit step function and $m_0 - \|z\|_1 = \varepsilon^3 > 0$ so that (13) is satisfied. Hence, for all $0 < \varepsilon < 1$, $M_\varepsilon \in \mathscr{M} \subset \mathscr{M}_{\text{odd}}$ is a Zames–Falb multiplier. Now, from the definition of the derivative matrix we have

$$\begin{aligned}N(x(t)) &= N(x_0 + \varepsilon y(t)) \\ &= N(x_0) + \varepsilon N'(x)|_{x_0} y(t) + g(t)\end{aligned}$$

where $\|g(t)\| = \mathrm{O}(\varepsilon^2)\|y(t)\|$. Using the fact that $y(t)$ is periodic with period $T \doteq 2\pi/\omega_0$, we obtain

$$
\begin{aligned}
&\lim_{\tau\to\infty} \tfrac{1}{\tau}\langle M_\varepsilon^* N(x_\tau), x_\tau\rangle_\tau \\
&= \lim_{\tau\to\infty} \tfrac{1}{\tau}\langle N(x_\tau), M_\varepsilon x_\tau\rangle_\tau \\
&= \lim_{\tau\to\infty} \tfrac{1}{\tau}\langle N(x), M_\varepsilon x\rangle_\tau \\
&= \frac{1}{\mathrm{T}}\langle N(x), M_\varepsilon x\rangle_T \\
&= \frac{1}{T}\left\langle N(x_0) + \varepsilon\frac{\partial N}{\partial x}(x_0)\,y + g,\ M_\varepsilon x_0 + \varepsilon M_\varepsilon y\right\rangle_T \\
&= \frac{1}{T}\langle N(x_0), M_\varepsilon x_0\rangle_T + \varepsilon\frac{1}{T}\langle N(x_0), M_\varepsilon y\rangle_T + \varepsilon\frac{1}{T}\langle (A_\mathrm{h}+A_\mathrm{s})\,y, M_\varepsilon x_0\rangle_T \\
&\quad + \varepsilon^2\frac{1}{T}\langle (A_\mathrm{h}+A_\mathrm{s})\,y, M_\varepsilon y\rangle_T + \frac{1}{T}\langle g, M_\varepsilon x_0\rangle_T + \frac{1}{T}\langle g, \varepsilon M_\varepsilon y\rangle_\mathrm{T} \\
&= 2\hat{m}_\varepsilon^*(0)\,x_0^\mathrm{T} N(x_0) + 0 \quad \text{(by orthogonality of sinusoids and constants)} \\
&\quad + 0 \quad \text{(by orthogonality of sinusoids and constants)} + \varepsilon^2 y_0^*\,\mathrm{herm}\,(\hat{m}_\varepsilon^*(\mathrm{j}\omega_0)(A_\mathrm{h}+A_\mathrm{s}))\,y_0 \\
&\quad + \mathrm{O}(\varepsilon^2)|\hat{m}_\varepsilon(\mathrm{j}0)|\,\|x_0\| + \mathrm{O}(\varepsilon^3)|\hat{m}_\varepsilon(\mathrm{j}0)| \\
&= \varepsilon^2 y_0^*\,\mathrm{herm}\,(\hat{m}_\varepsilon^*(\mathrm{j}\omega_0)(A_\mathrm{h}+A_\mathrm{s}))\,y_0 + O(\varepsilon^3)
\end{aligned}
\tag{24}
$$

where the last equality follows since $\hat{m}_\varepsilon(\mathrm{j}0) = \varepsilon^3$. Now, as $\varepsilon \to 0$, $\hat{m}_\varepsilon(\mathrm{j}\omega_0) \to \mathrm{j}\omega_0$; whence,

$$
\begin{aligned}
\lim_{\varepsilon\to 0} y_0^*\,\mathrm{herm}\,(\hat{m}_\varepsilon^*(\mathrm{j}\omega_0)(A_\mathrm{h}+A_\mathrm{s}))\,y_0 &= y_0^*(-\mathrm{j}\omega_0 A_\mathrm{s})\,y_0 \\
&= [1 \quad -j]\begin{bmatrix} 0 & \mathrm{j}\omega_0 a \\ -\mathrm{j}\omega_0 a & 0 \end{bmatrix}\begin{bmatrix} 1 \\ j \end{bmatrix} \\
&= -2\omega_0 a < 0
\end{aligned}
\tag{25}
$$

where the first equality follows by noting that the facts that A_h is Hermitian implies that jA_h is skew and that A_s is skew implies that $\mathrm{j}A_\mathrm{s}$ is Hermitian. Therefore,

$$
\begin{aligned}
\lim_{\tau\to\infty} \tfrac{1}{\tau}\langle M_\varepsilon^* N(x_\tau), x_\tau\rangle_\tau &= -2\varepsilon^2\omega_0 a + \mathrm{O}(\varepsilon^3) \\
&< 0 \quad \text{for all } \varepsilon > 0 \text{ sufficiently small.}
\end{aligned}
\tag{26}
$$
□

Remark 4

Theorem 1 completely characterizes the solution to Problem 1. It establishes that for positivity of a nonlinearity in $\mathscr{N}$ (or $\mathscr{N}_{\mathrm{odd}}$) to be preserved by every Zames–Falb multiplier in $\mathscr{M}$ (or $\mathscr{M}_{\mathrm{odd}}$), it is necessary and sufficient that the nonlinearity be the gradient of a convex function, a condition more restrictive than the incremental positivity condition (5).

Theorem 2

Suppose $M \in \mathscr{M}$ (or $M \in \mathscr{M}_{\mathrm{odd}}$). Then, M^*N is positive for all $N \in \mathscr{N}, N \in \mathscr{C}^1$ (or, resp. $N \in \mathscr{N}_{\mathrm{odd}}, N \in \mathscr{C}^1$) if, and only if,

$$\operatorname{Im}(\hat{m}(\mathrm{j}\omega)) = 0 \quad \text{for all } \omega. \tag{27}$$

Proof. (*if*) First, consider the case $M \in \mathscr{M}$, $N \in \mathscr{N}$, $N \in \mathscr{C}^1$. Suppose that (27) holds. It follows from (27) and the definition of the Fourier transform that $m(t) \doteq m_0 - z(t)$ must be an even function, i.e. $z(t) = z(-t)$, $\forall t$. Thus, for all $y \doteq Nx$,

$$\begin{aligned}
\langle M^*N(x), x\rangle &= \langle M^*y, x\rangle \\
&= \langle m_0^* y, x\rangle - \langle z^* * y, x\rangle \\
&= \operatorname{trace}[m_0 r_{yx}(0)] - \operatorname{trace}\left[\int_{-\infty}^{\infty} z(-t) r_{yx}(-t)\,\mathrm{d}t\right] \\
&= \operatorname{trace}[m_0 r_{yx}(0)] - \operatorname{trace}\left[\int_{-\infty}^{\infty} \tfrac{1}{2}(z(-t) + z(t)) r_{yx}(-t)\,\mathrm{d}t\right] \quad (\text{since } z(t) = z(-t)) \\
&= \operatorname{trace}[m_0 r_{yx}(0)] - \tfrac{1}{2}\operatorname{trace}\left[\int_{-\infty}^{\infty} z(t) r_{yx}(t)\,\mathrm{d}t\right] - \tfrac{1}{2}\operatorname{trace}\left[\int_{-\infty}^{\infty} z(t) r_{yx}(-t)\,\mathrm{d}t\right] \\
&= \operatorname{trace}[m_0 r_{yx}(0)] - 1/2\operatorname{trace}\left[\int_{-\infty}^{\infty} z(t)(r_{yx}(t) + r_{yx}(-t))\,\mathrm{d}t\right] \qquad (28) \\
&= \operatorname{trace}\left[\left(m_0 - \int_{-\infty}^{\infty} z(t)\,\mathrm{d}t\right) r_{yx}(0)\right] \\
&\quad + \tfrac{1}{2}\operatorname{trace}\left[\int_{-\infty}^{\infty} z(t)(2r_{yx}(0) - r_{yx}(t) - r_{yx}(-t))\,\mathrm{d}t\right] \\
&\geqslant 0 \qquad (29)
\end{aligned}$$

where that last inequality follows since, by (13), $m_0 - \int_{-\infty}^{\infty} z(t)\,\mathrm{d}t \geqslant 0$ and, by Lemma A3,

$$\operatorname{trace}[2r_{xy}(0) - r_{xy}(t) - r_{xy}(-t)] \geqslant 0 \ \forall t$$

When $M \in \mathscr{M}_{odd}$, $N \in \mathscr{N}_{\mathrm{odd}}$ and $N \in \mathscr{C}^1$, the result can be seen to hold by using arguments similar to the ones employed for that case in the proof of the (if) part of Theorem 1 in conjunction with (28).

(*Only if*) Noting that $\mathscr{N}_{\mathrm{odd}} \subset \mathscr{N}$, it suffices to find an instance of an $N \in \mathscr{N}_{\mathrm{odd}}$ for which M^*N is not positive whenever $\operatorname{Im}(\hat{m}(\mathrm{j}\omega)) \neq 0$. The counterexample given by (7) and (8) is one such instance. □

Remark 5

Theorem 2 completely characterizes the solution to Problem 2. It establishes that for a Zames–Falb multiplier to preserve positivity for every incrementally positive nonlinearity $N \in \mathscr{N}$ or $N \in \mathscr{N}_{\mathrm{odd}}$, it is necessary and sufficient to restrict the multiplier $\hat{m}(\mathrm{j}\omega)$ to be real-valued for all frequencies.

Remark 6

If the system in Figure 1 has a full-block feedback nonlinearity, Theorem 2 states rather trivially that the allowable multipliers M have the Fourier transform $\hat{m}(j\omega)$ where the scalar $\hat{m}(j\omega)$ is positive and real-valued. In this case, there is no distinction between positivity of the feedforward element H and that of MH. Therefore, incorporating such a multiplier does not reduce conservatism in the stability analysis and Theorem 2 is inconsequential.

However, if the system in Figure 1 has a block-diagonal feedback nonlinearity N i.e. $N \doteq \text{diag}(N_1, N_2, \ldots, N_m)$ where $N_i: \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$ satisfy (4)–(6), then Theorem 2 can be seen to be useful as follows. From 1, Lemma 3, it may be seen that every multiplier, say M_i, characterized by Theorem 2 to preserve positivity of the corresponding component full-block nonlinearity N_i has a (spectral) factorization of the form

$$M_i = M_{-i} M_{+i} = D_i^* D_i$$

where $D_i = M_{+i} = M_{-i}^*$, and both D_i and D_i^{-1} are stable. It follows that positivity of the composite nonlinearity $N \doteq \text{diag}(N_1, N_2, \ldots, N_m)$ is preserved by block-diagonal scaling

$$N \leftarrow DND^{-1}$$

where the frequency-response matrix $\hat{d}(j\omega)$ of D has the diagonal form as follows:

$$\hat{d}(j\omega) \doteq \text{diag}(\hat{d}_1(j\omega) I_1, \hat{d}_2(j\omega) I_2, \ldots, \hat{d}_n(j\omega) I_n)$$

where I_i denote $n_i \times n_i$ identity matrices and the scalars $\hat{d}_i(j\omega)$ denote the frequency responses of the spectral factors D_i. That is, for purposes of stability analysis via the equivalent system in Figure 2, we may take $M_-^* = M_+ = D$. Stability is then assured $MH \equiv D^* DH$ is positive or, equivalently, if the diagonally scaled linear operator DHD^{-1} is positive. Note that *in this case*, positivity of MH is *not* equivalent to that of H, which means that the real multipliers provided by Theorem 2 can reduce conservativeness in deciding stability of the system.

6. DISCUSSION

Multipliers in $\mathscr{M}(\tilde{\mathscr{M}})$ are positivity preserving for $\tilde{\mathscr{N}}(\mathscr{N})$ and, thus, define IQCs which are satisfied by those nonlinearities. Stability of the system can be ascertained using Theorem 1 of Reference [12]. We refer the reader to Reference [12] for a discussion of how IQCs are used in stability analysis. The contribution of this paper lies in a sharp characterization of such multipliers. The results obtained prescribe necessary and sufficient conditions. It may be noted that the conditions derived in [1] were sufficiency conditions only.

The reader will doubtless realize that the case of multiple scalar feedback nonlinearities, repeated or otherwise, which has been covered in the literature to date, implicitly assumes that the feedback nonlinearity is the gradient of a potential function. We believe that our Theorem 1 and Theorem 2 can be used with the results for the case of repeated scalar nonlinearities derived by D'Amato *et al.* [11] to derive those for the case of repeated block diagonal nonlinearities. A practical implementation of the Zames–Falb multipliers to solve the stability problem in the SISO setting amounts to solving an infinite dimensional linear program (see Reference [6]) which has been shown (see Reference [4]) to require typically less than 10 free elements owing to convexity of the problem. We speculate that a practical implementation in MIMO setting would lead to a linear matrix inequality problem which can be solved efficiently.

7. CONCLUSION

In their celebrated 1968 paper on nonlinear stability, Zames and Falb [1] determined a class of multipliers that preserve the incremental-positivity of norm-bounded, time-invariant, memoryless SISO nonlinearities. Results therein were proved only for the SISO nonlinearities and it was conjectured that the results might also be generalized to hold in the MIMO case. Validity of this conjecture has remained suspect, as is evident from the comments of Willems [13, Chapter 3, p. 66] as well as from the scarcity of literature on the analysis of such systems in MIMO setting even though the case of multiple scalar nonlinearities has received some attention (see Reference [9–11] and references therein). In this paper, we have demonstrated that their conjecture, as it stands, is incorrect and that for it to hold true, a further restriction needs to be applied. Specifically, it suffices either to restrict the MIMO nonlinearity to be the gradient of a convex real-valued function or to restrict the multiplier to be a real-valued function of frequency.

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APPENDIX A: BACKGROUND RESULTS

Stated below are a few essential background results.

Lemma A1

Let $G: L_2 \to L_2$ be a linear time-invariant operator having $n \times n$ frequency response matrix $\hat{g}(j\omega)$. Then

$$\langle Gx, x \rangle \geqslant 0 \quad \forall x \in L_2 \Leftrightarrow \lambda_{\min}(\text{herm}\,(\hat{g}(j\omega))) \geqslant 0 \quad \forall \omega.$$

Proof. This result follows directly via Parseval's theorem—see, for example, Reference [5].

$$\begin{aligned}\langle Gx, x \rangle &= \int_{-\infty}^{\infty} x^{\mathrm{T}}(t)\,[Gx](t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}^*(j\omega)\,\hat{g}(j\omega)\,\hat{x}(j\omega)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{x}^*(j\omega)\,\text{herm}\,(\hat{g}(j\omega))\,\hat{x}(j\omega)\,\mathrm{d}\omega.\end{aligned}$$

Whence, $\langle Gx, x \rangle \geqslant 0\, \forall x$ if, and only if, herm $(\hat{g}(j\omega))$ is positive semidefinite for all ω, which is equivalent to the condition $\lambda_{\min}(\text{herm}\,(\hat{g}(j\omega))) \geqslant 0\, \forall \omega$. □

Lemma A2

Let $N: \mathbb{R}^n \to \mathbb{R}^n$, $N \in \mathscr{C}^1$. Then,

$$(x-y)^{\mathrm{T}} N(x) + (y-x)^{\mathrm{T}} N(y) \geqslant 0 \quad \forall x, y \in \mathbb{R}^n \tag{A1}$$

if, and only if,

$$\operatorname{herm}(N'(x)) \geqslant 0 \quad \forall x \in \mathbb{R}^n. \tag{A2}$$

Proof. For all $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$, let $\zeta(\alpha) \doteq y + \alpha(x - y)$. Then, for all $x, y \in \mathbb{R}^n$, it holds that

$$\begin{aligned}(x-y)^{\mathrm{T}} N(x) + (y-x)^{\mathrm{T}} N(y) &= (x-y)^{\mathrm{T}}(N(x) - N(y)) \\ &= (x-y)^{\mathrm{T}} \int_y^x N'(\zeta)\,\mathrm{d}\zeta \\ &= (x-y)^{\mathrm{T}} \int_0^1 \left.\frac{\partial N(x)}{\partial x}\right|_{x=\zeta(\alpha)} \frac{\partial \zeta(\alpha)}{\partial \alpha}\,\mathrm{d}\alpha \\ &= (x-y)^{\mathrm{T}} \int_0^1 \operatorname{herm}\left(\left.\frac{\partial N(x)}{\partial x}\right|_{x=\zeta(\alpha)}\right)(x-y)\,\mathrm{d}\alpha \\ &= (x-y)^{\mathrm{T}} \left(\int_0^1 \operatorname{herm}(N'(x)|_{x=\zeta(\alpha)})\,\mathrm{d}\alpha\right)\mathrm{d}\alpha\,(x-y).\end{aligned}$$

Whence, the result follows. □

Lemma A3
Let $N: \mathbb{R}^n \to \mathbb{R}^n, N \in \mathscr{C}^1$ and $y = N(x)$. If $\operatorname{herm}(N'(x)) \geqslant 0\, \forall x \in \mathbb{R}^n$ then

$$\operatorname{trace}[2r_{yx}(0) - r_{yx}(\tau) - r_{yx}(-\tau)] \geqslant 0 \quad \forall \tau \in \mathbb{R}.$$

Proof. Suppose $\operatorname{herm}(N'(x)) \geqslant 0 \quad \forall x \in \mathbb{R}^n$. By Lemma A2,

$$(x(t) - x(t+\tau))^{\mathrm{T}} y(t) + (x(t+\tau) - x(t))^{\mathrm{T}} y(t+\tau) \geqslant 0 \quad \forall t$$

Integrating with respect to t, we have

$$\operatorname{trace}[r_{yx}(0)] - \operatorname{trace}[r_{yx}(-\tau)] + \operatorname{trace}[r_{yx}(0)] - \operatorname{trace}[r_{yx}(\tau)] \geqslant 0.$$ □

Lemma A4 (Chapter 5, p. 359)
Let $f: \mathbb{R}^n \to \mathbb{R}^n$, $f \in \mathscr{C}^1$, $x \in \mathbb{R}^n$. Then

$$P(x) \doteq \int_0^x f(x)\,\mathrm{d}x \tag{A3}$$

exists and is path independent if and only if

$$\operatorname{skew}(f'(x)) = 0 \quad \forall x \in R^n. \tag{A4}$$

Remark 7
Given a continuously differentiable vector field f, the above lemma gives a necessary and sufficiency condition for the existence of a potential function P. The condition (A4) generalizes the familiar condition $\operatorname{curl}(f) = 0$ satisfied by electric fields $f: \mathbb{R}^3 \to \mathbb{R}^3$ as a consequence of Maxwell's electromagnetic field equations [16].

APPENDIX B: NOTATION

$\mathbb{R}$	set of all real numbers
$\mathbb{C}$	set of all complex numbers
x, y	real signals — possibly vector-valued or matrix valued
F, G	operators
$\overline{(\cdot)}^{\mathrm{T}}$	conjugate transpose of vector or matrix $(\cdot)$
herm (m)	$= \frac{1}{2}(m + \bar{m}^{\mathrm{T}})$ for $m \in \mathbb{C}^{n \times n}$ or $\mathbb{R}^{n \times n}$
skew (m)	$= \frac{1}{2}(m - \bar{m}^{\mathrm{T}})$ for $m \in \mathbb{C}^{n \times n}$ or $\mathbb{R}^{n \times n}$
trace $[m]$	$= \sum_i m_{ii}$, (trace of a square matrix m)
$\lambda_{\min}(m)$	least eigenvalue of matrix m
P_τ	$[P_\tau x](t) = \begin{cases} x(t) & \text{if } -\tau \leqslant t \leqslant \tau \\ 0 & \text{otherwise} \end{cases}$ (two-sided time-truncation)
$\langle x, y \rangle$	$= \int_{-\infty}^{\infty} y(t)^{\mathrm{T}} x(t)\,\mathrm{d}t$
$\langle x, y \rangle_\tau$	$= \langle P_\tau x, P_\tau y \rangle = \int_{-\tau}^{\tau} y(t)^{\mathrm{T}} x(t)\,\mathrm{d}t$
$\lVert x \rVert_2$	$= \sqrt{\langle x, x \rangle}$
L_2	space of possibly vector valued signals x for which $\lVert x \rVert_2$ exists
$x(t) * y(t)$	$= \int_{-\infty}^{\infty} x(\tau) y^{\mathrm{T}}(t - \tau)\,\mathrm{d}\tau$ (convolution)
x^*	$x^*(t) = x^{\mathrm{T}}(-t)$ if x is a real signal
$r_{xy}(t)$	$= x * y^* = \int_{-\infty}^{\infty} x(t + \tau) y^{\mathrm{T}}(\tau)\,\mathrm{d}\tau$ (correlation function)
$\hat{x}$	$= \mathscr{F} x = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-j\omega t}\,\mathrm{d}t$ (Fourier transform)
$\lVert x \rVert_1$	$= \int_{-\infty}^{\infty} \lvert x(t) \rvert\,\mathrm{d}t$ (for scalar valued signals)
l_1	Space of signals x for which $\lVert x \rVert_1$ exists
$\mathscr{C}^1$	Set of continuous once-differentiable mappings
$N'(\zeta)$	Jacobian matrix $\partial N(\zeta)/\partial \zeta$ of $N : \mathbb{R}^n \to \mathbb{R}^m$, $N \in \mathscr{C}^1$

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