SAFE SWITCHING ADAPTIVE CONTROL:
STABILITY AND CONVERGENCE.

by

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Abstract

This thesis is concerned with the control of highly uncertain systems. A formal theoretical explanation of the model-mismatch instability problem often associated with adaptive control design schemes is proposed, and a generic solution is provided. A primary task of adaptive control is formulated as finding an asymptotically optimal, stabilizing controller, given the feasibility of adaptive control problem. Feasibility is defined as the existence of a stabilizing solution in a continuously parametrized candidate controller set. The multi-faceted adaptive control problem is thus placed in a setting of a standard optimization problem. The proposed solution, called safe adaptive control, robustly achieves this goal without any assumptions other than feasibility. Specifically, a list of the required plant-independent properties of the cost function is formulated. The result extends previous theoretical results in multiple model adaptive control, and provides a significant generalization by allowing the class of candidate controllers to have arbitrary cardinality, structure or dimensionality, and by strengthening the concept of tunability. The problem is motivated by a model-mismatch stability failure associated with a multitude of adaptive control schemes, exemplified by several simulation results.
Chapter 1

Introduction

1.1 Motivation

Control system design has historically been based on a model of the system to be controlled, with or without presupposed uncertainty bounds. When the system operates in a stationary or slowly-varying environment, or its parameters are slowly changing over time, a popular method of controlling the plant is adaptive control, using traditional continuously-tuned control algorithms and a single fixed or adaptive plant model. When the plant is fixed, and the modeling uncertainty is known to lie in some range, one resorts to robust control. However, circumstances often arise when neither of the methods generally described above guarantees success of the control operation. For instance, the plant (system to be controlled) is insufficiently known (poorly modeled process), its parameters change swiftly over time, and there are no reliably known bounds on the modeling uncertainty. In these cases, robust control may fail, as the real plant may fall outside presupposed uncertainty bounds around a nominal model, whereas adaptive continuous tuning can result in excessive transients, unacceptably poor performance or even instability. Adaptive control systems are nonlinear by their
nature, making the analysis more difficult, leading to the introduction of simplifying assumptions and approximations. Typically, adaptive theories achieve stability and convergence objectives by limiting attention to plants that satisfy restricting assumptions, e.g., the well known but difficult-to-satisfy standard assumptions of adaptive control [NA89]. Traditional pre-1980’s adaptive control theory was based on the following prior plant assumptions: linear time-invariance, minimum phase plant, no time delays, known upper bound of the plant order, known sign of the high-frequency gain, no noise, no external disturbances. While the use of these simplifications made mathematical analysis tractable, it became clear that these assumptions rarely hold in practice. The destabilizing effect of the arbitrarily small time delay on the robustness properties of the classical pre-1980’s adaptive theories was demonstrated in the well-known examples in [RVAS85]. The use of standard assumptions has been widely criticized. A number of modifications to the existing adaptive theories emerged, such as the inclusion of the dead-zone in the adaptive law, projection of estimated parameters onto a compact convex set in the parameter space, the use of $\sigma$-modification and $\varepsilon$-modification in the adaptive algorithms [IS96]. The resulting control designs were able to tolerate pre-specified amounts of time-delays or uncertainty. During the past two decades, a different distinct stream of improving robustness of adaptive control was oriented towards fuller utilization of the evolving measurement data to optimize performance and to perform necessary action if evidence in collected data suggests errors in assumed plant or uncertainty models. These recent advances in adaptive
control have encompassed *switching* among multiple models and/or controllers, such that the most troublesome assumptions on the plant, including those that had led to the model-mismatch failure with arbitrarily small time delay, are eliminated.

In this thesis, we address and solve the problem of model-mismatch instability. To begin with, we need a precise and crisp definition of adaptive control and its goals. With this in mind, recall the following beginning of the book “Adaptive control” by Åström and Wittenmark [ÅW95]:

“In everyday language, ‘to adapt’ means to change a behavior to conform to new circumstances. Intuitively, an adaptive controller is thus a controller that can modify its behavior in response to changes in the dynamics of the process and the character of the disturbances.”

Whether it is conventional, continuous adaptive tuning or more recent adaptive switching, adaptive control has an inherent property that it orders controllers based on evidence found in data. Any adaptive algorithm can thus be associated with a cost function, dependent on available data, that it minimizes, though this may not be explicitly present. The differences among adaptive algorithms arise in part due to the extent to specific algorithms employed to approximately compute cost-minimizing
controllers. And, major differences arise due to the extent to which additional assumptions are tied with this cost function. The cost function needs to be chosen to reflect control goals. The perspective adopted in this paper hinges on the notion of feasibility of adaptive control. An adaptive control problem is said to be feasible if the plant is stabilizable and at least one (a priori unknown) stabilizing controller exists in the candidate controller set that achieves the specified control goal for the given plant [ST97, Saf03]. Given feasibility, our view is that a primary goal of adaptive control is to recognize when the accumulated experimental data shows that a controller fails to achieve desired stability and performance objectives. If a destabilizing controller happens to be the currently active one, adaptive control should eventually switch it out of the loop, and replace it with an optimal, stabilizing one. An optimal controller is one that optimizes the controller ordering criterion (“cost function”) given the currently available evidence. This perspective renders the adaptive control problem in a form of a standard constrained optimization.

This formulation of the adaptive control task is, obviously, one aspect among a variety of ways of envisioning the goal of adaptive control, though it is clear that it is intertwined with them, rather than independent of them. Some other adaptive control problem formulations aim toward expanding of the candidate controller set on the fly [KI02]: other popular multiple-model methods base the stability/convergence proofs upon the premise of sufficient closeness of the uncertain plant to one of the candidate models.
It should be noted that we do not address in this thesis the issue of the dual control (in Feldbaum’s terminology [Fel60], compromise between investigating by means of persistent excitation in order to achieve a rapid and sure convergence to a stabilizing controller, and directing in order to obtain good quality control (e.g., ‘bumpless transfer’)).

1.2 Historical Perspective and Related Research

In this section, we give a short historical overview of the development of (robust) adaptive control and its present trends, as well as the limitations of the existing methods. We discuss the relative merit of the research proposed in this thesis.

Historically, alternative adaptive control methods have been developed in order to handle the situation where the available plant models are poor and unreliable (preventing pure robust control from achieving its goal), or the operating conditions are changing swiftly. To address the emerging need for robustness for larger uncertainties or achieve tighter performance specifications, an outer adaptive loop has been introduced (see, among others, [ZMF00, HLM03b, SWS04, ZMF01, HLM03a, CLS02]) that acts as a supervisory feedback loop that adjusts and changes the primary feedback controller. This outer loop performs its supervisory role by monitoring plant output data since
evidence found in it may lead to improvement of the primary active controller. A series of various implementations of this idea has been proposed in the past. Data-driven unfalsified adaptive control methods, introduced in [ST97] and underlying the work in this paper, exploit evidence in the plant output data to switch a controller out of the loop when the evidence proves that the controller is failing to achieve the stated goal.

Adaptive controllers are by their nature nonlinear, making mathematical analysis and design difficult. Thus, various simplifying assumptions and approximations have been introduced, from the fairly restrictive ones of the early, pre-1980’s adaptive control, to the much more relaxed, but still somewhat limiting assumptions of the modern switching control.

Several recent advances have involved the use of *multi-model controller switching* formulations of the adaptive control problem, *e.g.* supervisory based control design, reported in [Mor96, Mor97, KR96] and the related work of Morse and other authors. It was shown in [HLM03a] that supervisory control allows fast ’discontinuous’ adaptation in highly uncertain nonlinear systems, and thus leads to the improved performance and overcomes some limitations of classical adaptive control. These formulations have led to improved optimization-based adaptive control theories and, most importantly, significantly weaker assumptions of prior knowledge. Less
reliance on prior knowledge for stability proofs means more robust adaptive controllers. Both indirect ([Mor96, NB97, ZMF00, ZMF01] among others) and direct [FB86, Mår85, MP93, ST97] switching methods have been proposed for the adaptive supervisory loop. These methods have all been shown to admit proofs of convergence and stability under weakened assumptions. The indirect methods work by identifying a good approximation to the true plant from a set of candidate plant models; these indirect methods can be shown to be convergent if one assumes that the true system is sufficiently close to one of the candidate plant models. Direct methods do not require explicit identification of a plant model; instead these exploit real-time measurement data in an attempt to experimentally discover, via a process of elimination, whether a candidate controller is failing to meet performance and stability goals. The first truly direct switching methods, reported in [Mår85, FB86] are essentially plant-assumption-free, and are able to achieve global convergence to a stabilizing controller. However, they fail to fully use the information in the experimental data, and as a consequence are unduly slow in adaptation. The method of Safonov and Tsao [ST97] is, similarly, completely plant-model-free; and their algorithm is able to efficiently use information from measured data to evaluate potential performance levels of all candidate controllers simultaneously, without actually inserting them in the loop. As a result, the adaptation is much faster, and the convergence to a stabilizing controller occurs within a fraction of an unstable plant’s largest unstable time constant.
Following the work in [ST97], further progress was made in [SWS04] which clearly identified sufficient conditions for adaptive control to ensure stability and convergence to a controller that is robustly stabilizing and performing, provided that such a controller exists in the candidate controller pool (control problem feasibility). The theory was concerned with stability and finite time convergence for a finite candidate controller set.

1.3 Contribution

The contribution of this dissertation is the presentation of a parsimonious theoretical framework for examining stability and convergence of a switching adaptive control system using an infinite class of candidate controllers (typically, a continuum of controllers is considered). This property of the candidate controller class is essential when the uncertainties in the plant and/or external disturbances are so large that no set of finitely many controllers is likely to suffice in achieving the control goal. It is shown that, under some non-restrictive assumptions on the cost function (designer-based, not plant-dependent), stability of the closed loop switched system is assured, as well as the convergence to a stabilizing controller in finitely many steps. The framework is not restricted by the knowledge of noise and disturbance bounds, or any specifics of the unknown plant, other than the feasibility of the adaptive control problem.
1.4 Outline of the Dissertation

This dissertation is organized as follows.

- Chapter 2 presents an overview of the preliminary definitions and notation used.

- Chapter 3 poses the problem addressed in this thesis, and provides the theoretical results on stability and finiteness of switches for a general unknown plant. Then it exemplifies the construction of the cost function satisfying the required conditions.

- In Chapter 4, analysis for a time-varying nonlinear plant is given, and a specialization to the linear time-invariant case is provided.

- Chapter 5 presents simulation examples and counterexamples corroborating the theory.

- Chapter 6 provides discussion of the presented work and future directions.
Chapter 2

Basic Concepts

2.1 Preliminary Definitions and Notation

Let $\mathbf{Z}$ be the set of all possible output signals $z = [u, y]$ reproducible by the switching adaptive system $\Sigma : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$ in Figure 2.1, where $u, y$ are the measured plant control input and output vector signals, respectively. Let $z_{data} = [y_{data}, u_{data}] \in \mathbf{Z}$ represent the output signals recorded (hypothetically) in one single, infinite duration, experiment. At any time $\tau$, $P_\tau z_{data}$ is the actually available data obtained using the projection operator that truncates a signal after $t = \tau$.

Unless otherwise noted, it is assumed, throughout the paper, that all components of the system under consideration have zero-input zero-output property, so that when the system $\Sigma$ is undisturbed ($(r, d, n) = 0$), the pair $(y, u) = (0, 0)$ is an equilibrium solution.

We consider an infinite set $\mathbf{K}$ (e.g. containing a continuum) of candidate controllers. The finite controller set results will be derived as a special case. The parametrization of
K, denoted \( \Theta_K \), will initially be taken to be a subset of \( \mathbb{R}^n \); the more general case of infinite dimensional spaces will be discussed in Chapter 3.

**Definition 2.1.1** The adaptive control problem is said to be feasible if a candidate controller set \( K \) contains at least one controller that achieves stability and performance goals.

Comment 2.1.1 Feasibility is a necessary and sufficient condition for the existence of a stabilizing solution \( K \in K \) for the adaptive control problem as defined in Def. 2.1.1. Note that \( K \in K \) excludes control laws which do not converge to a particular stabilizing controller (e.g., dither control, which might achieve stability via fast oscillation between two or more destabilizing controllers).
Definition 2.1.2  The goal of adaptive control is finding an asymptotically optimal, stabilizing controller $K \in \mathbf{K}$, given the feasibility of the adaptive control problem.

Definition 2.1.3  A controller $K$ is said to be a feasible controller if it satisfies given stability and performance constraints.

Assumption 1  The adaptive control problem is feasible.

Comment 2.1.2  It is not known a priori which elements $K$ of the set $\mathbf{K}$ are feasible.

The following are the definitions of the proposed notions as well as some familiar definitions from the input/output stability theory [Saf80, DV75].

Definition 2.1.4  (Class $\mathbf{K}$) A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ belongs in class $\mathbf{K}$ (denoted $\phi \in \mathbf{K}$) if $\phi$ is continuous, strictly increasing and $\phi(0) = 0$.

Definition 2.1.5  Linear truncation operator $P_\tau : \mathbb{Z} \to \mathbb{Z}_\tau$ is given by

$$[P_\tau z](t) \overset{\Delta}{=} \begin{cases} z(t), & \text{if } 0 \leq t \leq \tau \\ \text{not defined otherwise.} & \end{cases}$$

The previous definition holds for any $\tau \in \mathbf{T}$ ($\mathbf{T}$ typically denotes time, so $\mathbf{T} = \mathbb{R}_+$ or $\mathbf{T} = \mathbb{Z}_+$ in discrete time systems).
For all $t \in T$, $P_\tau z$ is a truncated signal that resides in a normed space $Z_\tau$, which is related to $Z$ in the sense that $Z$ is an extended normed space induced by the collection of operators $\{P_\tau : \tau \in T\}$, and $Z = Z_\tau, \forall \tau \in T$. Notation $||z||_\tau$ refers to $||P_\tau z||$, where $||\cdot||$ is the norm defined on $Z$ or $Z_\tau$ (these norms need not be equal).

The $L_2$-norm of a truncated signal $P_\tau z$ is given as $||z||_\tau = \sqrt{\int_0^\tau ||z(t)||^2 dt}$, where $||z(t)||$ stands for the Euclidean norm of the vector $z$ at time $t$. The Euclidean norm of the parameterization $\theta_K \in \mathbb{R}^n$ of the controller $K$ is denoted $||\theta_K||$.

**Definition 2.1.6** The system $\Sigma : L_2e \rightarrow L_2e$ with input $w$ and output $z$ is said to be stable if there exists a function $\phi \in K$ such that $\forall w \in L_2e, w \neq 0$: $||z||_\tau \leq \phi(||w||_\tau), \forall \tau \in \mathbb{R}_+$

Otherwise, $\Sigma$ is said to be unstable. If $\phi$ exists and is linear, $\Sigma$ is said to be finite-gain stable.

Specializing to the system in Figure 2.1, stability of the closed loop system $\Sigma$ means $\sup_{\tau \in \mathbb{R}_+} ||[y, u]||_\tau \leq \phi(\sup_{\tau \in \mathbb{R}_+} ||r||_\tau)$, for some $\phi \in K$ and $\forall r \in L_2e, r \neq 0$.

**Definition 2.1.7** The signal $z_{\text{data}} = [y_{\text{data}}, u_{\text{data}}] \in Z$ is the particular system output data signal obtained in one infinite duration experiment, and $P_\tau z_{\text{data}}$ is the actually available data at the time $\tau$. 

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Definition 2.1.8 ([ST97]). For every $K \in K$, a fictitious reference signal $\tilde{r}_K(z_{data}, \tau)$ is defined to be an element of the set

$$\tilde{R}_K(z_{data}, \tau) \doteq \{ r | K \begin{bmatrix} r \\ y \end{bmatrix} = u, P_{\tau} z_{data} = \begin{bmatrix} P_{\tau} u \\ P_{\tau} y \end{bmatrix} \}.$$ 

In other words, $\tilde{r}_K(z_{data}, \tau)$ is a hypothetical reference signal that would have exactly reproduced the measured data $z_{data}$ had the controller $K$ been in the loop for the entire time period over which the data $z_{data}$ was collected.

Comment 2.1.3 If the fictitious reference signal can be expressed as $\tilde{r}_K = K^{-1} u + y$, with $K$ representing the operator rather than controller parametrization, i.e. if the candidate controllers have “causally-left-invertible” structure [ST97], then there exists a unique $\tilde{r}_K$ for each $K$. Otherwise, $\tilde{r}_K \in \tilde{R}_K(z_{data}, \tau) \doteq \{ r | K \begin{bmatrix} r \\ y \end{bmatrix} = u, P_{\tau} z_{data} = \begin{bmatrix} P_{\tau} u \\ P_{\tau} y \end{bmatrix} \}.$

Definition 2.1.9 Stability of the system $\Sigma : w \mapsto z$ is said to be falsified by the data $(w, z)$ if

$$\sup_{\tau \in \mathbb{R}_+, \|w\|_{\tau} \neq 0} \frac{\|z\|_{\tau}}{\|w\|_{\tau}} = \infty.$$ 

Otherwise, it is said to be unfalsified.
Definition 2.1.10  The cost functional \( V(K, z, t) \) is a causal-in-time mapping
\[
V : K \times Z \times T \rightarrow \mathbb{R}_+ \cup \{\infty\}
\]

Definition 2.1.11  [Ber99] A functional \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be coercive if \( \lim f(x) = \infty \) when \( ||x|| \to \infty, x \in \mathbb{R}^n \).  

Definition 2.1.12  The true cost \( V_{\text{true}} : K \rightarrow \mathbb{R}_+ \cup \{\infty\} \) is defined as \( V_{\text{true}}(K) = \sup_{z \in Z, \tau \in T} V(K, z, \tau) \).

Definition 2.1.13  A robust optimal controller \( K_{RSP} \) is one that stabilizes (in the sense of the Def. 2.1.6) the given plant and minimizes the true cost \( V_{\text{true}} \).

Therefore, \( K_{RSP} = \arg \min_{K \in K} (V_{\text{true}}(K)) \) (and is not necessarily unique). Due to the feasibility assumption, at least one such \( K_{RSP} \) exists, and \( V_{\text{true}}(K_{RSP}) < \infty \).

Definition 2.1.14  A sequence \( \{t_k\}_{k=1}^{\bar{N}}, \bar{N} \in \mathbb{N} \cup \{\infty\} \) is an ordered sequence of times \( (t_{k+1} > t_k, \forall k = 1, 2, ..., \bar{N}) \).

In conjunction with the algorithm A1 that will be introduced shortly, the cost functional \( V(K, z, t) \), and the particular \( z_{\text{data}} \) from one experiment, \( \{t_k\}_{k=1}^{\bar{N}}, \bar{N} \in \mathbb{N} \cup \{\infty\} \) denotes the ordered sequence of switching times. \( K_k \) denotes the controller switched in the loop at time \( t_k, k = 1, ..., \bar{N} \), that remains in the loop until some time \( t_{k+1} > t_k \), when \( K_{k+1} \) is switched in the loop according to Algorithm A1. \( \hat{K}_t \) is the currently active controller at time \( t \). Thus, \( \hat{K}_t = K_k \) on \( t \in [t_k, t_{k+1}) \).
For simplicity, we will consider the case of no noise and disturbances in the following definition.

**Definition 2.1.15** Given the switched adaptive system \( \Sigma : \omega \to z \) in Figure 2.1, the cost function \( V(K, z, t) \) and a candidate controller set \( \mathbf{K} \), we say that the pair \((V, \mathbf{K})\) is cost detectable if, for each \( \hat{K}_t \) that eventually converges after finitely many switches to a controller \( K_N \in \mathbf{K} \) \((K_N = \lim_{t \to \infty} \hat{K}_t) \in \mathbf{K})\), it holds that the following two conditions are equivalent:

- a). \((\omega, z)\) falsifies stability of the switched system
- b). \(\lim_{t \to \infty} V(K_N, z, t) = \infty\)

**Comment 2.1.4** Cost detectability is plant-independent. It holds without regard to plant assumptions or ‘priors’. It depends only on the cost function \( V \) and the class of controllers \( \mathbf{K} \). In this respect, it is stronger than the ‘tunability’ concept of Morse et. al. [MMG92] which is similar, but depends on additional assumptions (i.e., ‘priors’) on the plant – assumptions which vary from one application to the next, and which not hold in practice or may otherwise be difficult to verify.

**Definition 2.1.16**

\[
V = \{ V_{z,t} : z \in Z, t \in T \} : \mathbf{K} \to \mathbb{R}_+ \tag{2.1}
\]

is a family of functionals with the common domain \( \mathbf{K} \), with \( V_{z,t}(K) \doteq V(K, z, t) \).
**Definition 2.1.17** A level set in $\mathbb{R}^n$ is defined as $L(\alpha) \triangleq \{ x \in \mathbb{R}^n | f(x) \leq \alpha \}$ for some $\alpha \in \mathbb{R}$. 

**Definition 2.1.18** The set $L \triangledown \{ K \in \mathcal{K} | V_{z,t_0}(K) \leq V_{true}(K_{RSP}), V \in \mathcal{V} \}$ is the level set in the controller space corresponding to the cost at the first switching time instant. 

**Definition 2.1.19** [Rud76]. If $E \subset X$, and $f$ is a function defined on $X$, the restriction of $f$ to $E$ is the function $g$ whose domain of definition is $E$ such that $g(p) = f(p)$ for $p \in E$. 

With the family of functionals $\mathcal{V}$ with a common domain $\mathcal{K}$, a restriction to the set $L \subseteq \mathcal{K}$ is associated, defined as a family of functionals $\mathcal{W} \triangledown \{ W_{z,t}(K) : z \in Z, t \in T \}$ with a common domain $L$. 

Consider now the cost minimization hysteresis switching algorithm reported in [MMG92], together with the cost functional $V(K, z, t)$. Recall that the data signal vector that appears in the cost functional $V$ is obtained by the time truncation of $z$, i.e. $P_\tau z_{data}$ for the particular $z_{data}$ obtained in one experiment. The algorithm returns, at each time instant $\tau$, a controller $\hat{K}_\tau$ which is the active controller in the loop.
1. Initialize: Let $t = 0$, $\tau = 0$; choose $\varepsilon > 0$.

Let $\hat{K}_t \in K$ be the first controller in the loop.

2. $\tau \leftarrow \tau + 1$.

If $V(\hat{K}_t, z, \tau) > \min_{K \in K} V(K, z, \tau) + \varepsilon$ then

$t \leftarrow \tau$ and $\hat{K}_t \leftarrow \arg \min_{K \in K} V(K, z, \tau)$.

3. $\hat{K}_\tau \leftarrow \hat{K}_t$;

4. go to 2.

In the above algorithm, $t$ is the time of the last controller switch. The switch occurs only when the current unfalsified cost related to the currently active controller exceeds the minimum (over $K$) of the current unfalsified cost by at least $\varepsilon$, as shown in Figure 2.2. Here, the hysteresis step $\varepsilon$ serves to limit the number of switches on any finite time interval to a finite number, and so prevents the possibility of limit cycle type of instability (‘chattering’). It also ensures a non-zero dwell time between switches.

The above algorithm has its origins in [MMG92] (see also [MGHM88]); however, a theorem developed here clearly articulates optimal steps to be followed in designing adaptive control law selectors that efficiently and reliably overcome the model-mismatch stability problem. The hysteresis switching lemma of [MMG92] implies that a switched
sequence of controllers $K_{t_k}$ ($k = 1, 2, \ldots$) that minimize over $K$ the current unfalsified cost $V(K, z, t)$ at each switch-time $t_k$ will also stabilize if the cost for each fixed controller $K$ has the properties that $1^\circ$ it is a monotone increasing function of time and $2^\circ$ it is uniformly bounded above if and only if $K$ is stabilizing. But, these properties of adaptive methods were demonstrated in [MMG92] only by introducing unnecessary assumptions on the plant.

**Definition 2.1.20** [WZ77]. Let $S$ be a topological space. A family $\mathcal{F} \doteq \{f_\alpha : \alpha \in A\}$ of complex functionals with a common domain $S$ is said to be equicontinuous at a point $x \in S$ if for every $\epsilon > 0$ there exists an open neighborhood $N(x)$ such that $\forall y \in N(x)$, $\forall \alpha \in A$, $|f_\alpha(x) - f_\alpha(y)| < \epsilon$. The family is said to be equicontinuous on $S$ if it is
equicontinuous at each \( x \in S \). \( F \) is said to be uniformly equicontinuous on \( S \) if \( \forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0 \) such that \( \forall x, y \in S, \forall \alpha \in A, y \in N_\delta(x) \Rightarrow |f_\alpha(x) - f_\alpha(y)| < \epsilon \), where \( N_\delta \) denotes an open neighborhood of size \( \delta \).

In a metric space \( S \) with a metric \( d_S \), uniform equicontinuity means that \( \forall x, y \in S, \forall \alpha \in A, d_S(x, y) < \delta \Rightarrow |f_\alpha(x) - f_\alpha(y)| < \epsilon \).

Lemma 2.1.1 If \( (S, d) \) is a compact metric space, then any family \( F = \{f_\alpha : \alpha \in A\} \) that is equicontinuous on \( S \) is uniformly equicontinuous on \( S \).

Proof: The proof is a simple extension of the theorem that asserts uniform continuity property of a continuous mapping from a compact metric space to another metric space. Let \( \epsilon > 0 \) be given. Since \( F = \{f_\alpha : \alpha \in A\} \) is equicontinuous on \( S \), we can associate to each point \( p \in S \) a positive number \( \phi(p) \) such that \( q \in S, d_S(p, q) < \phi(p) \Rightarrow |f_\alpha(p) - f_\alpha(q)| < \frac{\epsilon}{2} \) for all \( \alpha \in A \).

Let \( J(p) = \{q \in S | d_S(p, q) < \frac{1}{2} \phi(p)\} \). Since \( p \in J(p) \), the collection of all sets \( J(p), p \in S \) is an open cover of \( S \); and since \( S \) is compact, there is a finite set of points \( p_1, \ldots, p_n \in S \) such that \( S \subseteq \bigcup_{i=1}^{n} J(p_i) \). Let us set \( \delta = \frac{1}{2} \min_{1 \leq i \leq n}[\phi(p_i)] \). Then \( \delta > 0 \), because a minimum of a finite set of positive numbers is positive, as opposed to the \( \inf \) of an infinite set of positive numbers which may be 0. Now let \( p, q \in S, d_S(p, q) < \delta \). Then, \( p \in J(p_m), m \in 1, \ldots, n \). Hence, \( d_S(p, p_m) < \frac{1}{2} \phi(p_m) \) and \( d_S(q, p_m) < d_S(q, p) + d_S(p, p_m) < \delta + \frac{1}{2} \phi(p_m) < \phi(p_m) \). Thus, \( |f_\alpha(p) - f_\alpha(q)| < \frac{\epsilon}{2} \) for all \( \alpha \in A \).
\[ |f_\alpha(p) - f_\alpha(p_m)| + |f_\alpha(p_m) - f_\alpha(q)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon \forall \alpha \in A. \] Since \( \delta \) holds for all \( p, q \in S \) and all \( \alpha \in A \), \( \mathcal{F} \) is uniformly equicontinuous.
Chapter 3

Results

3.1 Problem Formulation

Åström and Wittenmark begin their book “Adaptive control” [ÅW95] in the following way: “In everyday language, ‘to adapt’ means to change a behavior to conform to new circumstances. Intuitively, an adaptive controller is thus a controller that can modify its behavior in response to changes in the dynamics of the process and the character of the disturbances.”

Whether it is conventional, continuous adaptive tuning or more recent adaptive switching, adaptive control has an inherent property that it orders controllers based on evidence found in data. Any adaptive algorithm can thus be associated with a cost function, dependent on available data, that it minimizes, though this may not be explicitly present. The differences among adaptive algorithms arise in part due to specific algorithms employed to approximately compute cost-minimizing controllers. And, major differences arise due to the extent to which additional assumptions are tied with this cost function. The cost function needs to be chosen to reflect control
goals. The perspective adopted in this thesis hinges on the notion of feasibility of adaptive control. Given feasibility, our view is that a primary goal of adaptive control is to recognize when the accumulated experimental data shows that a controller fails to achieve desired stability and performance objectives. If a destabilizing controller happens to be the currently active one, adaptive control should eventually switch it out of the loop, and replace it with an optimal, stabilizing one. An optimal controller is one that optimizes the controller ordering criterion (“cost function”) given the currently available evidence. This perspective renders the adaptive control problem in a form of a standard constrained optimization.

This formulation of the adaptive control task is, obviously, one aspect among a variety of ways of envisioning the goal of adaptive control, though it is clear that it is intertwined with them, rather than independent of them. Some other adaptive control problem formulations aim toward expanding of the candidate controller set on the fly [KI02]. We do not address in this work the issue of the dual control (in Feldbaum’s terminology [Fel60], compromise between investigating by means of persistent excitation in order to achieve a rapid and sure convergence to a stabilizing controller, and directing in order to obtain good quality control (‘bumpless transfer’)).

The results of this work widen the theoretical ground established previously by allowing the class of candidate controllers to be infinite [SPS05]. This property of the candidate
controller class is essential when the uncertainties are so large that no set of finitely many controllers is likely to suffice in achieving the control goal. It is shown that, under some mild additional assumptions on the cost function (designer-based, not plant-dependent), stability of the closed loop switched system is assured, as well as the convergence to a stabilizing controller in finitely many steps.

The problem of adaptive control is formulated as follows:

**Definition 3.1.1** The goal of adaptive control is finding an asymptotically optimal, stabilizing controller, given the feasibility of the adaptive control problem.

3.2 Main Result

The main results on stability and finiteness of switches are developed in the sequel.

**Lemma 3.2.1** Consider the feedback adaptive control system $\Sigma$ in Figure 2.1 with input $w$ and output $z = [u, y]$, together with the hysteresis switching algorithm A1. Suppose there are finitely many switches, and that $\forall K \in K$ the corresponding fictitious reference signal (FRS) generator has a stable structure. If the adaptive control problem is feasible (Def. 2.1.1), and the associated cost functional $V(K, z, t)$ is continuous in time and the following properties are satisfied:

- $(V, K)$ is cost-detectable (Def. 2.1.15)
\( V \) is monotone increasing in time

then the last switched controller is stabilizing, and moreover, the system response \( z(t) \) with the final controller satisfies the performance inequality

\[
\sup_{\tau \in \mathcal{T}, z \in \mathcal{Z}} V(\hat{K}_\tau, z, \tau) \leq V_{\text{true}}(K_{\text{RSP}}) \pm \epsilon, \text{ for all } \tau, z.
\]

**Proof:** Let the controller switched in the loop at time \( t_i \) be denoted \( K_i \); and \( i \in I = \{0, 1, ..., N\}, N \in \mathbb{N} \cup \{\infty\} \) be the indices of the switching instants. Suppose that stability of the closed loop system \( \Sigma \) with the first controller in the loop \( K_0 \), switched at \( t_0 \), is falsified by the data \((\hat{r}_{K_0}, \tilde{z}_{\text{data}})\); i.e.

\[
\max_{\hat{r}_{K_0} \in \tilde{R}_{K_0}(z, \tau)} \sup_{\tau \in \mathbb{R}^+} \|\tilde{z}_{\text{data}}\|_\tau = \infty
\]

Due to the cost detectability property:

\[
\lim_{\tau \to \infty} V(K_0, \tilde{z}_{\text{data}}, \tau) = \infty
\]

In particular, since the cost is monotone in time, \( \exists t_1 > t_0 \) such that

\[
V(K_0, \tilde{z}_{\text{data}}, t_1) = \varepsilon + \min_{K} V(K, \tilde{z}_{\text{data}}, t_1) = V(K_1, \tilde{z}_{\text{data}}, t_1)
\] (3.1)
whereby the switching adaptive algorithm replaces $K_0$ with $K_1$ in the loop. Again, if stability of $\Sigma$ with $K_1$ in the loop is falsified by $(\tilde{r}_{K_1}, z_{data})$, then at some $t_2 > t_1$

$$V(K_1, z_{data}, t_2) = \varepsilon + \min_K V(K, z_{data}, t_2)$$

Denote the last switching time instant $t_N$ (at this point, we do not discuss the finiteness of $N$; Lemma 3.2.3 demonstrates that $N \in \mathbb{N}$ under additional assumptions on the cost functional). We show here that $K_N$ is stabilizing.

Assume that at some time $\tau^*$ the switched controller in the loop is $K_k$, $k \in I \setminus \{0\}$ (therefore, $t_0 < \tau^*$). If stability of $K_k$ is falsified by $(\tilde{r}_{K_k}, z_{data})$, then again we switch to $K_{k+1}$ (and so, $\tau^* < t_N$). Suppose that $K_k$ is the last switched controller in the loop ($k = N$). Then

$$\forall t \in T, \ V(K_k, z_{data}, t) < \varepsilon + \min_K V(K, z_{data}, t)$$

$$< \varepsilon + V_{true}(K_{RSP})$$

$$< \infty$$

due to the uniform upper-boundedness of $V(K, z, t)$ by the true cost $V_{true} : K \rightarrow \mathbb{R}_+ \cup \{\infty\}$ for all plant data (Def. 2.1.12), and feasibility of the control problem (Def. 2.1.1).
Due to the cost detectability, stability of $\Sigma$ with $K_N$ is not falsified by $(\tilde{r}_K, z_{\text{data}})$, that is,

$$\sup_{\tau \in \mathbb{R}^+} \frac{||z_{\text{data}}||_\tau}{||\tilde{r}_K||_\tau} < \infty \quad (3.3)$$

Since fictitious reference signal generators have stable structure, the ratio $\frac{||\tilde{r}_K||_\tau}{||r||_\tau}$ is finite for all $\tau \in \mathbb{R}^+$ (Lemma A.0.1 in the Appendix). This, along with (3.3) implies

$$\sup_{\tau \in \mathbb{R}^+} \frac{||z_{\text{data}}||_\tau}{||r||_\tau} < \infty.$$  

□

Lemma 3.2.2 Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous and coercive function on $\mathbb{R}^n$. Then for any scalar $\alpha \in \mathbb{R}$, the level set $L(\alpha) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ is compact.

Proof: Since $L(\alpha) \subset \mathbb{R}^n$, we show that $L(\alpha)$ is closed and bounded: Let $\{x_m\} \subseteq L(\alpha)$ be a convergent sequence, and $\bar{x} = \lim_{m \rightarrow \infty} x_m$. Since $f$ is continuous, $f(\bar{x}) = \lim_{m \rightarrow \infty} f(x_m)$. Also, $f(x_m) \leq \alpha$, $\forall m \in \mathbb{N}$. Then, $f(\bar{x}) = \lim_{m \rightarrow \infty} f(x_m) \leq \lim_{m \rightarrow \infty} \alpha = \alpha$, so $\bar{x} \in L(\alpha)$. Hence, $L(\alpha)$ is closed. To show that is $L(\alpha)$ is bounded, proceed by contradiction. Assume that $L(\alpha)$ is not bounded; then there exists a sequence $\{y_m\} \subseteq L(\alpha)$ such that $\lim_{m \rightarrow \infty} ||y_m|| = \infty$. Since $f$ is coercive, $\lim_{m \rightarrow \infty} f(y_m) = \infty$; in particular, $\exists N \in \mathbb{N}$ such that $\forall k \geq N f(y_k) > \alpha$, for any fixed $\alpha \in \mathbb{R}$. Then, $\{y_m\} \not\subseteq L(\alpha)$, which contradicts the above assumption. Thus, $L(\alpha)$ is closed and bounded in $\mathbb{R}^n$, therefore compact. □
Lemma 3.2.3 Consider the feedback adaptive control system in Figure 2.1, together with the switching algorithm A1. If the adaptive control problem is feasible (Def. 2.1.1), and the associated cost functional $V(K, z, t)$ is cost-detectable and monotone increasing in time and, in addition,

- For all $\tau \in T, z \in Z$, the cost functional $V(K, z, t)$ is coercive on $K \subseteq \mathbb{R}^n$ (i.e. $\lim_{||K|| \to \infty} V(K, z, \tau) = \infty$), and

- The family $\mathcal{W} = \{W_{z,t}(K) : z \in Z, t \in T\}$ of restricted cost functionals with a common domain $L$ is equicontinuous on $L$,

then the number of switches is uniformly bounded above for all $z \in Z$ by some $\bar{N} \in \mathbb{N}$.

Proof: Due to Lemma 3.2.2, the level set $L$ is compact. Then, the family $\mathcal{W} = \{W_{z,t}(K) : z \in Z, t \in T\}$ is uniformly equicontinuous on $L$ (see Lemma 2.1.1), i.e. for a hysteresis step $\epsilon$, $\exists \delta > 0$ such that for all $z \in Z, t \in T, K_1, K_2 \in L, ||K_1 - K_2|| < 2\delta \Rightarrow |W_{z,t}(K_1) - W_{z,t}(K_2)| < \epsilon$ (i.e. $\delta = \delta(\epsilon)$ is common to all $K \in L$ and all $z \in Z, t \in T$). Since $L$ is compact, there exists a finite open cover $\mathcal{C}_N = \{B_\delta(K_i)\}_{i=1}^N$, with $K_i \in \mathbb{R}^n, i = 1, \ldots, N$ such that $L \subseteq \bigcup_{i=1}^N B_\delta(K_i)$, where $N$ depends on the chosen hysteresis step $\epsilon$ (this is a direct consequence of the definition of a compact set). Let $\hat{K}_{t_j}$ be the controller switched into the loop at the time $t_j$, and the corresponding minimum cost achieved is $\hat{V} = \min_{K \in K} V(K, z, t_j)$. Consider that at the time $t_{j+1} > t_j$ a switch occurs at the same cost level $\hat{V}$, i.e. $\hat{V} = \min_{K \in K} V(K, z, t_{j+1})$.
where $V(\hat{K}_{t_j}, z, t_{j+1}) > \min_{K \in \mathcal{K}} V(K, z, t_{j+1}) + \epsilon$. Therefore, $\hat{K}_{t_j}$ is falsified, and so are all the controllers $K \in B_{2\delta}(\hat{K}_{t_j})$. Let $I_j$ be the index set of the as-yet-unfalsified $\delta$-balls of controllers at the time $t_j$. Since $\hat{K}_{t_j} \in B_{\delta}(K_i)$, for some $i \in \bar{I} \subset I_j^1$, also falsified are all the controllers $K \in B_{\delta}(K_i) \supset \hat{K}_{t_j}$, so that $I_{j+1} = I_j \setminus \{i\}$, i.e. $I_j$ is updated according to the following algorithm ($j$ is the index of the switching time $t_j$):

**Unfalsified index set algorithm:**

1. **Initialize:** Let $j = 0$, $I_0 = \{1, \ldots, N\}$.

2. $j \leftarrow j + 1$. If $I_{j-1} = \emptyset$: Set $I_j = \{1, \ldots, N\}$ // Optimal cost increases
   Else
   
   $I_j = I_{j-1} \setminus \{i\}$, where $i \in I_{j-1}$ is such that $B_{\delta}(K_i) \supset \hat{K}_{t_{j-1}}$.

3. go to (2);

Therefore, the number of possible switches to a single cost level is upper-bounded by $N$, the number of $\delta$-balls in the cover of $L$. The next switch (the very first after $N^{th}$ one), if any, must occur to a cost level higher than $\tilde{V}$, due to the monotonicity of $V$. But then, according to Algorithm 1, $|V(\hat{K}_{t_j+N+1}, z, t_{j+N+1}) - \tilde{V}| > \epsilon$, with $d(\hat{K}_{t_j+N+1}, \tilde{K}_{t_k}) < 2\delta$.

---

1 $\bar{I}$ is not necessarily a singleton as $\hat{K}_{t_j}$ may belong to more than one balls $B_{\delta}(K_i)$, but it suffices for the proof that there is at least one such index $i$.
\[ j \leq k \leq j + N \text{ and } V(\tilde{K}_{tk}, z, t_k) = \tilde{V}. \]

Combining the two bounds, the overall number of switches is upper-bounded by:

\[ \bar{N} = N \frac{V_{\text{true}}(K_{\text{RSP}}) - \min_{K \in K} V(K, z, 0)}{\epsilon} \]

The finite controller set case is obtained as a special case of the Lemma 3.2.3, with \( N \) being the number of candidate controllers instead of the number of \( \delta \)-balls in the cover of \( L \). The main result follows.

**Theorem 3.2.1** Consider the feedback adaptive control system \( \Sigma \) in Figure 2.1, together with the hysteresis switching algorithm A1. Suppose that the adaptive control problem is feasible (Def. 2.1.1), and the associated cost functional \( V(K, z, t) \) is continuous in time, the pair \( (V, K) \) satisfies the cost detectability property, and the conditions of Lemma 3.2.3 hold. Then, the system is stable. Moreover, for each \( z \), the system converges after finitely many switches to controller \( K_N \) that satisfies the performance inequality

\[ V(K_N, z, \tau) \leq V_{\text{true}}(K_{\text{RSP}}) + \epsilon \text{ for all } \tau. \tag{3.4} \]

**Proof:** Invoking Lemma 3.2.3 proves that there are finitely many switches. Then, Lemma 3.2.1 shows that the adaptive controller stabilizes and that (3.4) holds. \( \square \)
Comment 3.2.1  Note that, due to the coerciveness of $V$, $\min_{K \in \mathcal{K}} V(K, z, 0)$ is bounded below (by a nonnegative number, if the range of $V$ is a subset of $\mathbb{R}_+$), for all $z \in \mathcal{Z}$.

Comment 3.2.2  The parametrization of the set of candidate controllers can be more general than $\Theta_K \subseteq \mathbb{R}^n$; in fact, it can belong to an arbitrary infinite dimensional space; however $K$ has to be compact in that case, in order to ensure uniform equi-continuity property.

Note that the switching ceases after finitely many steps for all $z \in \mathcal{Z}$. If the system input is sufficiently rich so as to increase the cost more than $\epsilon$ above the level at the time of the latest switch, a switch to a new controller that minimizes the current cost will eventually occur at some later time. The values of these cost minima at any time are monotone increasing and bounded above by $V_{true}(K_{RSP})$. Thus, sufficient richness of the system input (external reference signal, disturbance or noise signals) will affect the cost to approach $V_{true}(K_{RSP}) \pm \epsilon$.

Technical remark 3.2.1  The minimization of the cost functional over the infinite set $K$ is tractable if the compact set $K$ can be represented as a finite union of convex sets, i.e. the cost minimization is a convex programming problem.

The following lemma identifies the goals of adaptive control that fail to be guaranteed when one of the conditions of Theorem 3.2.1 ceases to hold.
Lemma 3.2.4 The conditions of Theorem 3.2.1 are optimal in the sense that, if at least one of them is not satisfied, stability of the switched system and/or boundedness of the switches are not guaranteed.

Proof: We will construct the proof by considering a failure of one condition at a time. 1) Assume that the conditions of Theorem 3.2.1 are satisfied except for cost detectability. Then, if stability of the closed-loop system with the current controller (at time $t_1$) $K_1$ in the loop is falsified by the available data $[u_{data}, y_{data}]$, then for some $\tilde{r}_{K_1}$,

$$\sup_{t \in R^+} \frac{|| (y_{data}, u_{data}) ||_t}{\phi(|| \tilde{r}_{K_1} ||_t)} = \infty$$

If cost-detectability property of $V$ does not hold, then

$$\sup_{t \in R^+} V(K_1, [u_{data}, y_{data}], t) \leq C_1 < \infty$$

for some constant $C_1$, and in particular, it may happen that

$$\sup_{t \in R^+} V(K_1, [u_{data}, y_{data}], t) < V_{true}(K_{RSP}) + \varepsilon$$

Thus, a switch to another controller in AI will never occur, despite the instability evidence found in data. 2) Assume that the conditions of Theorem 3.2.1 are
satisfied except for monotonicity in time of the cost function. Suppose we have exhausted all the switches \((N, \text{the number of } \delta\text{-balls in the cover of } \mathbf{L})\) to a single cost level \(\tilde{V}\), as defined in Lemma 3.2.3. The next, \((N + 1)^{th}\) switch, if any, may occur to a cost level lower than \(\tilde{V} = \min_k V(K, z, t_k), k \in \{j, \ldots, j + N\}\), that is, \(V(\hat{K}_{t_{j+N}}, z, t_{j+N+1}) \geq \frac{\varepsilon}{2} + \min_k V(K, z, t_{j+N+1})\), but \(\min_k V(K, z, t_{j+N+1}) < V(\hat{K}_{t_k}, z, t_k), k \in \{j, \ldots, j + N\}\). After we have exhausted all possible switches to the new cost level \(\tilde{V}_{\text{new}} = \min_k V(K, z, t_{j+N+1})\), we can now switch to a level \(\frac{\varepsilon}{2}\) higher than \(\tilde{V}_{\text{new}}\), but less than \(\tilde{V}\). By oscillating around \(\tilde{V}\), the number of switches can grow unbounded.

\[\square\]

3.3 Cost Function Example

An example of the cost function and the conditions under which it ensures stability and finiteness of switches according to Theorem 3.2.1 may be constructed as follows. Consider (a not necessarily zero-input zero-output) system \(\Sigma : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}\) in Figure 2.1. Choose as a cost functional:

\[
V(K, z, t) = \max_{\tau \leq t} \frac{||y||^2 + ||u||^2}{||\tilde{r}_k||^2 + \alpha} + \beta + \gamma ||K||^2
\]  

(3.5)

where \(\alpha, \beta, \gamma\) are arbitrary positive numbers. \(\alpha\) is used in order to prevent \(V = 0\) when \(\tilde{r} = y = u = 0\) (unless \(\Sigma\) has zero-input zero-output property), \(\beta\) ensures \(V > 0\) even when \(||K|| \equiv 0\), and \(\gamma\) scales the importance of \(||K||^2\), where \(||K||\) is the Euclidean
norm of the controller parametrization (recall that $K \in K \subseteq \mathbb{R}^n$). Also, $|| \cdot ||_t$ stands for
the truncated 2-norm: $||x||_t = \sqrt{\int_0^t |x(\tau)|^2 d\tau}$. Such a cost function satisfies the three
required properties of Theorem 3.2.1.

**Comment 3.3.1** Equation (3.5) has this form provided the candidate controllers are
causally-left-invertible (otherwise, $V(K, z, t) = \max_{\hat{r}_K \in \hat{H}_K(z)} \{\text{RHS of (3.5)}\}$).

### 3.3.1 Stability Verification

Recall that the controller switched in the loop at time $t_i$ is denoted $K_i$; and
$i \in I = \{0, 1, ..., N\}$, $N \in \mathbb{N} \cup \{\infty\}$ are the indices of the switching instants. When
$t_0 = 0$, we have $\forall K \in K$, $V(K, z, t_0) = \beta + \gamma ||K||^2 > 0$. Let the controller switched
at time $t_i$ be denoted $\hat{K}_i$. Then, due to the cost minimization property of the switching
algorithm, $K_0 = \arg \min_K V(K, z, 0)$, and $V(K_0, z, 0) = \beta + \gamma ||K_0||^2 > 0$.

Denote by $t = t_N$ the time of the last switch (we will invoke later Lemma 3.2.3 to show
that $N$ is finite), and the corresponding controller $K_N$. Consider the time interval $[0, t_1)$.
During this time period, the active controller in the loop is $\hat{K}(t) = K_0$.

\[
V(K_0, z, t^-_1) = V(K_0, z, t_1) = \max_{\tau \leq t_1} \frac{||y||^2_\tau + ||u||^2_\tau}{||\hat{r}_{K_0}||^2_\tau + \alpha} + \beta + \gamma ||K_0||^2
\]

\[
= \max_{\tau \leq t_1} \frac{||y||^2_\tau + ||u||^2_\tau}{||r||^2_\tau + \alpha} + \beta + \gamma ||K_0||^2
\]
since $\tilde{r}_{K_0} \equiv \hat{r}(K_0, z, t) \equiv r(t), t \in [0, t_1)$. The equality in (3.6) is due to the continuity of $V$ in time.

Since $r$ is uniformly bounded, $||r||^2_t = \int_0^t |r(\tau)|^2 d\tau < \infty$.

At $t = t_1$, the cost of the current controller exceeds the current minimum by $\epsilon$:

$$V(K_0, z, t_1) = \max_{\tau \leq t_1} \frac{||y||^2_{\tau} + ||u||^2_{\tau}}{||\tilde{r}_{K_0}||^2_{\tau} + \alpha + \gamma||K_0||^2}$$

$$= \epsilon + \min_K V(K, z, t_1)$$

(3.7)

and so, according to the hysteresis switching algorithm, a switch occurs to the controller $K_1 = \arg \min_K V(K, z, t_1)$. Expression in (3.7) is finite since $\epsilon$ is finite and

$$\min_K V(K, z, t_1) \leq \sup_{t \in T, z \in Z} \min_K V(K, z, t) \equiv V_{true}(K_{RSP})$$

where $V_{true}(K_{RSP})$ is finite due to the feasibility assumption. Denoting the sum $\epsilon + \min_K V(K, z, t_1)$ by $\psi_1$, we have:

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Now consider the next switching period, \([t_1, t_2)\). The active controller in the loop is \(\hat{K}(t) = K_1\). We have:

\[
0 < \beta + \gamma \|K_1\|^2 \leq V(K_1, z, t_1) = \min_{\hat{K}} V(K, z, t_1)
\]

\[
= \max_{\tau \leq t_1} \frac{||y||^2 + ||u||^2}{||\hat{K}_1||^2 + \alpha} + \beta + \gamma \|K_1\|^2 < \infty
\]

where the second inequality from the left follows from the monotone increasing property of \(V\).

Therefore, \(\|\hat{r}_{K_1}\|_{t_1} < \infty\).
Now,

\[ \infty > \epsilon + \min_K V(K, z, t_2) = V(K_1, z, t_2) \]

\[ \geq \max_{\tau \leq t_2} \frac{||y||_\tau^2 + ||u||_\tau^2}{||\tilde{r}_K||_\tau^2 + \alpha} + \beta + \gamma ||K_1||^2 \]

\[ \geq \beta + \gamma ||K_1||^2 > 0 \]

Thus, $||\tilde{r}_K||_{t_2}$ is finite, and so are $||y||_{t_2}, ||u||_{t_2}$.

By induction, we conclude that

\[ ||y||_{t_N} \leq \infty, \quad ||u||_{t_N} \leq \infty \quad (3.8) \]

where $t_N$ is the last switching time.

Since $t_N$ was the last switching time:

\[ 0 < \beta + \gamma ||K_N||^2 \leq V(K_N, z, t) = \]

\[ \max_{\tau \leq t} \frac{||y||_\tau^2 + ||u||_\tau^2}{||\tilde{r}_K||_\tau^2 + \alpha} + \beta + \gamma ||K_N||^2 \]

\[ < \epsilon + \min_K V(K, z, t), \quad \forall t \geq t_N \]
Thus, $||\hat{r}_{K_N}||_t$ is finite for any finite $t > t_N$.

$$0 < \beta + ||K_N||^2 \leq \sup_{t \in \mathbb{R}^+} V(K_N, z, t) = \sup_{t \in \mathbb{R}^+} \max_{\tau \leq t} \frac{||y||^2 + ||u||^2}{||\hat{r}_{K_N}||^2 + \alpha} + \beta + \gamma ||K_N||^2$$

$$< \epsilon + \sup_{t \in \mathbb{R}^+} \min_{K} V(K, z, t) \leq \epsilon + V_{true}(K_{RSP}) < \infty \ \forall t$$

From here we conclude that stability of the closed loop switched system with the last controller $K_N$ is unfalsified by $(\hat{r}_{K_N}, z)$. Invoking Lemma A.0.1 in Appendix, we obtain stability result.

### 3.3.2 Finiteness of Switches Verification

First, $V(K, z, t)$ is clearly coercive on $K$ due to the addition of the term $\gamma ||K||^2$. As for the equicontinuity of the family $\mathcal{W}$ of restricted cost functionals on $L$, it is assured if one can demonstrate equicontinuity of the family $\mathcal{V} = \{V_{z,t}(K) : z \in Z, t \in T\}$ on $K$. That is, we look for the conditions under which $V(K', z, t)$ is continuous on $K$ for each fixed $z \in Z, t \in T$ (note: continuity of $V$ on $K$ is sought, not uniform continuity), in such a way that for every $\epsilon > 0$ there exists an open neighborhood $N(x)$ for each $x \in K$ such that $\forall y \in N(x), \forall z \in Z, t \in T, |V_{z,t}(x) - V_{z,t}(y)| < \epsilon$ (i.e. the size of the ball $N(x)$ is the same $\forall z \in Z, t \in T$). Thus, considering (3.5), we have the following remark:
Comment 3.3.2 If the cost function is given by (3.5), the conditions of the Theorem 3.2.3 for the finiteness of switches are satisfied if the fictitious signal generator $\tilde{r}$ is continuous in the controller parametrization.

For a simple gain-type of the controller (of 1-DOF structure), this would mean that one should choose the candidate controller set as $K \subseteq \mathbb{R}^n \setminus \{0\}$ since $\tilde{r}_K = K^{-1}u + y$ is not continuous for $K = 0$. 
Chapter 4

Extensions of the main result

4.1 Switching Control of Time-Varying Plants

Adaptive switching scheme discussed so far provides guarantees of stability and convergence for a general nonlinear plant (with arbitrary but bounded noise and/or disturbances), whose unknown dynamics are time-invariant. In other words, the adaptive algorithm aims toward finding (given sufficient excitation) a robust controller - one that stabilizes the time-invariant plant regardless of the disturbances. To have a ‘truly’ adaptive control system, one that is able to track changes in a plant whose parameters are either slowly time-varying or subject to infrequent large jumps (e.g. component-failure-induced), one needs to deemphasize importance of the old data, which may be not demonstrative of the current plant behavior. For slow parameter variations, one usually endows a cost function (control selection law, in general) with data windowing, or fading memory. For instance, the cost functional to be minimized usually has an integral term:

\[ J(\varphi, t) = ||\varphi(t)||^2 + \int_0^t e^{-\lambda(t-\tau)} \cdot ||\varphi(\tau)||^2 d\tau \]  

(4.1)
where $\lambda$ is a small non-negative number (‘forgetting factor’), $\varphi$ may be a vector of output data, identification error [NB97] etc. In such situations, convergence to a particular robustly stabilizing controller (given sufficient excitation) is neither achieved nor sought after. The property of the cost function that is lost by data windowing is its monotonicity in time. As a consequence, we do not have uniform shrinking of the candidate controller set anymore; recall that at each time instant, $\varepsilon$-cost minimization hysteresis algorithm falsifies a subset of the original candidate controller set whose current unfalsified cost level exceeds $V_{\text{true}}(K_{RSP})$; the resulting unfalsified controller sets form a nested, uniformly in time shrinking set, non-empty due to the feasibility assumption. Discarding time monotonicity, previously falsified controllers may be selected as optimal ones. Guarantees of convergence and finiteness of switches on the time interval $[0, \infty)$ are lost, but stability is preserved under the modified definition of stability. The type of instability induced by infinitely fast switching can be avoided if an arbitrary, bounded away from zero, positive ratcheting step $\varepsilon$ is used.

In the following, we introduce several definitions pertaining to the frozen-time analysis of stability of slowly time-varying plants, similarly as in the [ZW91].

**Definition 4.1.1**  *The unknown plant $P$ whose parameters are frozen at their values at time $t^*$ is denoted $P^{t^*}$; the closed loop switched system $\Sigma$ with the plant $P^{t^*}$ is denoted $\Sigma^{t^*}$. The set of all possible output signals $z = [u, y]$ reproducible by the switching system $\Sigma^{t^*}$ is denoted $Z^{t^*}$.  \diamond*
**Definition 4.1.2** A controller $K$ is said to be a feasible controller for $\Sigma^{t^*}$ if it satisfies given performance and stability constraints for $\Sigma^{t^*}$.

**Definition 4.1.3** The adaptive control problem associated with the switched system $\Sigma$ in Figure 2.1 is said to be feasible if there exists at least one feasible controller for each parameter variation of the plant $P$, i.e. for all $\Sigma^{t^*}$, $t^* \in T$.

**Assumption 2** The adaptive control problem associated with the switched system $\Sigma$ in Figure 2.1 is feasible.

**Comment 4.1.1** It is not known a priori which controllers $K$ in the set $K$ are feasible.

**Definition 4.1.4** Stability of a system $\Sigma^{t^*} : w \mapsto z$ is said to be falsified by data $(w, z)$ if

$$\sup_{\tau \in \mathbb{R}_+, \|w\|_\tau \neq 0} \frac{\|z\|_\tau}{\|w\|_\tau} = \infty$$

Otherwise, it is said to be unfalsified.

**Definition 4.1.5** The closed loop switched system $\Sigma^{t^*}$ is associated with the true cost $V_{true}^{\Sigma^{t^*}} : K \rightarrow \mathbb{R}_+ \cup \{\infty\}$, defined as $V_{true}^{\Sigma^{t^*}}(K) = \sup_{z \in \mathbb{Z}^{t^*}, \tau \geq t^*} V(K, z, \tau)$

**Definition 4.1.6** For $\Sigma^{t^*}$, a robust optimal controller $K^{\Sigma^{t^*}}_{RSP}$ is a feasible controller that minimizes the true cost $V_{true}^{\Sigma^{t^*}}$.

Due to the feasibility assumption, at least one such $K^{\Sigma^{t^*}}_{RSP}$ exists $\forall t^* \in T$, and $V_{true}^{\Sigma^{t^*}}(K^{\Sigma^{t^*}}_{RSP}) < \infty$. 

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**Definition 4.1.7** Given a switched adaptive system $\Sigma^{t^*} : \omega \rightarrow z$ in Figure 2.1, the cost function $V(K, z, t)$ and candidate controller set $K$, we say that the pair $(V, K)$ is cost detectable if, for each $\hat{K}_t$ that eventually converges after finitely many switches to a controller $K_N \in K$ ($K_N = (\lim_{t \rightarrow \infty} \hat{K}_t) \in K$), it holds that the following two conditions are equivalent:

a). $(\omega, z)$ falsifies stability of the switched system

b). $\lim_{t \rightarrow \infty} V(K_N, z, t) = \infty$

\[\diamond\]

**Lemma 4.1.1** Consider the switched zero-input zero-output system $\Sigma : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e}$ in Figure 2.1 where the unknown plant is slowly time-varying. The input to the system is $w = [r \ d \ x_0]^T$ (the fictitious input corresponding to $K$ is $\tilde{w}_K = [\tilde{r}_K \ \tilde{d}_K \ \tilde{x}_0]^T \in \tilde{W}_K(z)$) and the output is $z = [u \ y]^T$. Given a cost function of the type

$$V(K, z, t) = \max_{\tilde{w}_K \in \tilde{W}_K(z)} \left\{ \int_0^t e^{-\lambda_1(t-\tau)}||z(\tau)||^2 d\tau \right\} + \beta + \gamma||K||^2, \text{ if } t > 0$$

$$\beta + \gamma||K||^2, \text{ if } t = 0$$

where $\lambda_2 \geq \lambda_1 > 0$ are the weights on the past values of input and output data, respectively (‘forgetting factors’), and $\beta, \gamma$ are arbitrary positive constants as described
in Section 3.3. Then, the safe adaptive switching controller stabilizes the system \( \Sigma \).

**Proof:** If stability of the system \( \Sigma \) with \( K \) in the loop is falsified, then for some \( \tilde{w}_K \in \tilde{W}_K(z) \) there does not exist \( \phi \in K \) such that \( \limsup_{t \to \infty} ||z||_t \leq \phi(\limsup_{t \to \infty} ||\tilde{w}_K||_t) \), where \( ||z||_t = ||(u, y)||_t \doteq \sqrt{||u||^2_t + ||y||^2_t} \), and \( ||\zeta||_t \doteq \sqrt{\int_0^t e^{\lambda_1 \tau} ||\zeta(\tau)||^2 d\tau} \) is \( e^{\lambda_2} \)-weighted, \( L_2 \)-induced norm of a signal \( \zeta(t) \) (in general, norm weight for the output need not coincide with the norm weight for the input signal). In particular:

\[
\begin{align*}
\infty &= \limsup_{t \to \infty} \frac{||z||^2_t}{||\tilde{w}_K||^2_t} = \limsup_{t \to \infty} \frac{\int_0^t e^{\lambda_1 \tau} ||z(\tau)||^2 d\tau}{\int_0^t e^{\lambda_2 \tau} ||\tilde{w}_K(\tau)||^2 d\tau} \\
&\leq \limsup_{t \to \infty} e^{(\lambda_2-\lambda_1)t} \frac{\int_0^t e^{\lambda_1 \tau} ||z(\tau)||^2 d\tau}{\int_0^t e^{\lambda_2 \tau} ||\tilde{w}_K(\tau)||^2 d\tau} \\
&= \limsup_{t \to \infty} \frac{\int_0^t e^{-\lambda_1 (t-\tau)} ||z(\tau)||^2 d\tau}{\int_0^t e^{-\lambda_2 (t-\tau)} ||\tilde{w}_K(\tau)||^2 d\tau} \\
&< \limsup_{t \to \infty} \max_{\tilde{w}_K \in \tilde{W}_K(z)} \frac{\int_0^t e^{-\lambda_1 (t-\tau)} ||z(\tau)||^2 d\tau}{\int_0^t e^{-\lambda_2 (t-\tau)} ||\tilde{w}_K(\tau)||^2 d\tau} + \beta + \gamma ||K||^2 \\
&= \limsup_{t \to \infty} V(K, z, t)
\end{align*}
\]

From the last equation, cost detectability holds. Since the plant is time-varying, the switching will never stop, in general. Consider the time \( t \) between any two switches, i.e. \( t_k < t < t_{k+1} \). Let the currently active controller at time \( t \) be \( \hat{K}_t \). Then, \( V(\hat{K}_t, z_{data}, t) < \varepsilon + \min_K V(K, z_{data}, t) < \varepsilon + V_{true}(\hat{K}_{RSP}) \), where
\[ V_{\text{true}}(K_{\text{RSP}}) \triangleq \min_K \sup_{z \in Z_{\tau}^{\Sigma}, \tau \geq t} V^{\Sigma'_i}(K, z, \tau), \] which is finite by feasibility assumption. Thus, \( V(\hat{K}_t, z_{\text{data}}, t) < \infty, \forall t \in T. \) Due to cost detectability, stability of the closed loop switched system is unfalsified, which, along with the result in Lemma A.0.1 implies \( \sup_{t \in \mathbb{R}_+} \frac{\|z_{\text{data}}\|}{\|r\|} < \infty. \)

In this case, since the unknown plant is varying in time, or the operating conditions are changing, the switching may never stop. However, the number of switches on any finite time interval can be derived. To this end, we use the result of Hespanha et al. in [HLM03b] with some modifications. Recall from the algorithm A1 that a currently active controller \( \hat{K} \) will be switched out of the loop at time \( t \) if its associated cost satisfies \( V(\hat{K}, z, t) \geq \varepsilon + \min_K V(K, z, t). \) Similarly as in [HLM03b], we observe that this switching criterion will not be affected if we multiply both of its sides by some positive function of time \( \Theta(t): \)

\[ \Theta(t)V(\hat{K}, z, t) \geq \varepsilon \Theta(t) + \Theta(t) \min_K V(K, z, t) \]

Though we use the non-monotone in time cost function \( V \) as in (6.2) in the actual switching algorithm, for analysis purposes we use its scaled version \( V_{m}(K, z, t) \triangleq \Theta(t)V(K, z, t) \), where the positive function of time \( \Theta(t) \in L_{2e} \) is chosen so as to make
$V_m(K, z, t)$ monotone increasing in time. The step 2) in the switching algorithm A1 will also be modified to:

$$2) \quad V(\hat{K}_t, z, \tau) \geq \varepsilon + \min_{1 \leq i \leq N} \min_{K \in B_i^d} V(K, z, \tau)$$

where $B_i^d$ is the $i^{th}$ $\delta$-ball in the finite cover of $L$, and $N$ is their number (Lemma 3.2.3). This yields a hierarchical hysteresis switching similar to the one proposed in [HLM03b].

**Lemma 4.1.2** Consider the system $\Sigma : \mathcal{L}_{2\varepsilon} \rightarrow \mathcal{L}_{2\varepsilon}$ from the preceding sections, the cost (4.2), and an infinite candidate controller set $K$. Suppose that the switching algorithm is chosen to be the hierarchical additive hysteresis switching. Then, the number of switches $\mathcal{N}_{(t_0, t)}$ on any finite time interval $(t_0, t)$, $0 \leq t_0 < t < \infty$ is bounded above as:

$$\mathcal{N}_{(t_0, t)} \leq 1 + N + \frac{N}{\varepsilon \Theta(t)} (V_m(K, z, t) - \min_{K \in K} V_m(K, z, t_0))$$

**Proof:** For brevity, we will omit $z_{data}$ from $V(K, z_{data}, t)$ in the sequel. Suppose that $K_{t_k}$ is switched in the loop at time $t_k$, and remains there until time $t_{k+1}$. Then,

$$V_m(K_{t_k}, t) \leq \varepsilon \Theta(t) + V_m(K, t) \quad \forall t \in [t_k, t_{k+1}], \forall K \in K$$
Since $K_{t_k}$ was switched at time $t_k$, we also have

$$V_m(K_{t_k}, t_k) \leq V_m(K, t_k) \quad \forall K \in K$$

Due to continuity of $V$ in time,

$$V_m(K_{t_k}, t_{k+1}) = \varepsilon \Theta(t) + V_m(K_{t_{k+1}}, t_{k+1})$$

Now consider the non-trivial situation when more than one controller is switched in the loop on the time interval $(t_0, t)$. Then, some $K_q \in K$ must be active in the loop at least $\nu \geq \frac{N(t_{k_i+1}) - 1}{N}$ times ($N$ is the number of $\delta$-balls in the cover of $L$). Denote the time intervals when $K_q$ is active in the loop as

$$[t_{k_1}, t_{k_1+1}), [t_{k_2}, t_{k_2+1}), \ldots, [t_{k_\nu}, t_{k_\nu+1})$$

Due to the properties of the switching algorithm and monotonicity of $V_m$ we have:

$$V_m(K_q, t_{k_i+1}) = \varepsilon \Theta(t) + V_m(K_{t_{k_i+1}}, t_{k_i+1})$$

$$i \in \{1, 2, \ldots, \nu - 1\}$$

$$\geq \varepsilon \Theta(t) + V_m(K_{t_{k_i+1}}, t_{k_i})$$

$$\geq \varepsilon \Theta(t) + V_m(K_q, t_{k_i})$$
Also, because the switching time intervals are nonoverlapping,

\[ V_m(K_q, t_{k+1}) \geq V_m(K_q, t_{k+1}) \]

and so

\[ \varepsilon \Theta(t) + V_m(K_q, t_{k_i}) \leq V_m(K_q, t_{k_{i+1}}) \]

Since this holds \( \forall i \in \{1, 2, ..., \nu - 1\} \), we obtain

\[ (\nu - 1)\varepsilon \Theta(t) + V_m(K_q, t_{k_1}) \leq V_m(K_q, t_{k_{\nu}}) \]

\[ \Rightarrow (\nu - 1)\varepsilon \Theta(t) + V_m(K_q, t_0) \leq V_m(K, t) \]

\[ \forall K \in \mathcal{K} \]

Therefore, since \( \mathcal{K}(t_0, t) \leq \nu N + 1 \), we derive

\[ \mathcal{K}(t_0, t) \leq 1 + N + \frac{N}{\varepsilon \Theta(t)}(V_m(K, z, t) - \min_{K \in \mathcal{K}} V_m(K, z, t_0)) \]

\[ \forall K \in \mathcal{K} \]

which is finite since \( \varepsilon > 0 \), \( \Theta(t) \) is a positive, \( \mathcal{L}_2e \) function of time, and \( \min_{K \in \mathcal{K}} V_m(K, z, t_0) \) is finite due to feasibility assumption, as well as \( V_m(K, z, t) \) for some \( K \in \mathcal{K} \).
4.2 Stability Analysis in the LTI Setting

The above results can be specialized to the case of an LTI plant. In particular, one can derive an explicit bound for the state of the switched system when the last switched controller is the robustly stabilizing and performing controller $K^\ast$. Let the unknown plant $P$ in Fig. 2.1 be an LTI plant with control input $u$ and measured output $y$. In addition to a piecewise continuous bounded reference signal $r$, it is assumed that an unknown bounded disturbance $d$ and noise $n$ are acting at the plant input and output, respectively. A set of candidate controllers $K$ is considered, which can be an arbitrary infinite set of controllers (of LTI structure in the setting studied in this section). As before, the existence of $K^\ast = K_{RSP} \in K$ is assumed, such that $K^\ast$ robustly stabilizes $P$ for any bounded disturbance $d$ and noise $n$ at the plant input and output, respectively.

Denote the minimal state space representation of the unknown plant $P$ as $(A_p, B_p, C_p, D_p)$. Let state-space representation of an individual candidate controller $K$ be denoted as $(A_k, B_k, C_k, D_k)$. Then, $\rho(A_s) < 0$ for the closed loop state transition matrix $A_s$:

$$
A_s = \begin{bmatrix} 
A_{s1} & A_{s2} \\
A_{s3} & A_{s4} 
\end{bmatrix}
$$
where

\[ A_{s1} = A_p - B_p(I + D_kD_p)^{-1}D_k C_p \]
\[ A_{s2} = B_p(I + D_kD_p)^{-1}C_k \]

etc.

As before, \( \{t_i\}_{i \in \mathcal{I}} \) is an ordered sequence of switching times for some \( \mathcal{I} \subseteq \mathbb{N} \cup \{\infty\} \)

\( t_{i+1} > t_i, \forall i \in \mathcal{I} \). \( K_i \) is the controller switched in the loop at time \( t_i, i \in \mathcal{I} \), whereas \( \hat{K}(t) \) is the currently active switched controller at time \( t \):

\[ \hat{K}(t) = K_i, \forall t \in [t_i, t_{i+1}). \]

Denote the state space realization of \( \hat{K}(t) \) by \( (\hat{A}(t), \hat{B}(t), \hat{C}(t), \hat{D}(t)) \), such that, between switching times, \( \hat{A}(t) = A_i, \forall t \in [t_i, t_{i+1}) \) (similarly for other state-space matrices), where \( t_i \) denotes the time instant when \( K_i \) is switched in the loop.

Let the minimal state space realization for the plant \( \mathcal{P} \) be written as:

\[ \dot{x}_p = A_p x_p + B_p u \]
\[ y = C_p x_p + D_p u \]
The dynamic equations for the switched controller $\hat{K}(t)$ can be written as:

$$\dot{x} = \hat{A}(t)\hat{x} + \hat{B}(t)(r - y - n)$$
$$\nu = \hat{C}(t)\hat{x} + \hat{D}(t)(r - y - n)$$

where $u(t) = \nu(t) + d(t)$ (for simplicity of exposition, 1-DOF structure of the controller is assumed). The dynamic equations for the piecewise-LTI interconnected system can be written as:

$$\dot{x} = \mathcal{A}(t)x + \mathcal{B}(t)\omega$$
$$y = \mathcal{C}(t)x + \mathcal{D}(t)\omega$$

where $\omega = \begin{bmatrix} d \\ r - n \end{bmatrix}$, $x = \begin{bmatrix} x_p \\ \hat{x} \end{bmatrix}$, and

$$\mathcal{A}(t) = \begin{bmatrix} \mathcal{A}(t)_1 & \mathcal{A}(t)_2 \\ \mathcal{A}(t)_3 & \mathcal{A}(t)_4 \end{bmatrix}$$

with

$$\mathcal{A}(t)_1 \doteq A_p - B_p(I + \hat{D}(t)D_p)^{-1}\hat{D}(t)C_p$$
$$\mathcal{A}(t)_2 \doteq B_p(I + \hat{D}(t)D_p)^{-1}\hat{C}(t)$$

$$\mathcal{B}(t) = \begin{bmatrix} \mathcal{B}(t)_1 & \mathcal{B}(t)_2 \\ \mathcal{B}(t)_3 & \mathcal{B}(t)_4 \end{bmatrix}$$
with

\[ B(t)_1 = B_p (I + \dot{D}(t) D_p)^{-1} \]

\[ B(t)_2 = B_p (I + \dot{D}(t) D_p)^{-1} \dot{D}(t) \quad \text{etc.} \]

and similarly for \( C(t) \), \( D(t) \).

Note that \( \dot{x} \) is differentiable, if:

- the state of the previously active controller is retained as the initial state of the newly switched controller (due to the requirement of bumpless switching, needed for smooth performance), and

- the states of individual controllers are differentiable in time.

When the last switched controller is the robustly stabilizing and performing controller \( K_* \), we can derive explicit bound for the state of the switched system. For \( t \geq t_N \),

\[ \dot{K}(t) = K_N = K_* \], the RSP controller. The behavior of the switched system (4.2-4.3) is then described by constant matrices \( \mathcal{A}_*, \mathcal{B}_* \equiv \begin{bmatrix} B_p & -B_p D_{k_*} \\ 0 & B_{k_*} \end{bmatrix} \) and \( \mathcal{C}_* \equiv \begin{bmatrix} C_p \ 0 \end{bmatrix} \) (assuming \( D_p = 0 \)). The state transition matrix of the closed loop system (4.2-4.3) is \( \Phi(t, t_{k_*}) = e^{\mathcal{A}_*(t-t_{k_*})} \). Due to the exponential stabilizability of \( \mathcal{P} \) by \( K_* \), we
have \(|\|e^{A_*(t-t_{k^*})}\|| \leq c e^{-\lambda(t-t_{k^*})}\), for some positive constants \(c, \lambda\). Applying variation of constants formula to the state of the switched system \(x\), we obtain:

\[
\|x(t)\| \leq \|e^{A_*(t-t_{k^*})}\||x(t_{k^*})\| + \int_{t_{k^*}}^{t} \|e^{A_*(t-\tau)}B_\omega\||\omega||_\infty d\tau
\]

\[
\leq c e^{-\lambda(t-t_{k^*})}\|x(t_{k^*})\| + B_\epsilon \frac{C}{\lambda}(1 - e^{\lambda(t-t_{k^*})})\||\omega||_\infty
\]

\[
\leq c e^{-\lambda(t-t_{k^*})}\|x(t_{k^*})\| + B_\epsilon \frac{C}{\lambda}\||\omega||_\infty
\]

From (3.8), \(|\|C_\epsilon x(t_{k^*})\|| \leq \tilde{\Phi}_{k^*}\), and so \(|\|x(t_{k^*})\|| \leq \tilde{\Phi}_{k^*} < \infty\). Thus, \(|\|x(t)\|| is bounded for all \(t\), and

\[
\lim_{t \to \infty} \|x(t)\| \leq B_\epsilon \frac{C}{\lambda}\||\omega||_\infty
\]
Chapter 5

Examples

5.1 Simulation Example

When describing algorithm A1 in Chapter 2 we said that it originated as the hysteresis
switching algorithm in [MMG92]. We emphasized that the power of the hysteresis
switching lemma was clouded in the cited work by imposing unnecessary assumptions
on the plant in the demonstrations of the algorithm functionality. One of the plant
properties required in [MMG92] for ensuring cost detectability was the minimum
phaseness of the plant. We have shown in theory that the cost detectability is assured by
a proper choice of a cost function, and is not dependent on the plant or exogenous sig-
nals. In the following, we present a simulation example that demonstrates these findings.

Assume that the true, unknown plant is linear time-invariant with the transfer function
given by $G^*(s) = \frac{s-1}{s(s+1)}$. It is desired that the output of the plant behaves as the output
of the stable, minimum phase reference model $G_{ref} = \frac{1}{s+1}$. Presumed given is the set
of three candidate controllers: $C_1(s) = -\frac{s+1}{s+2.6}$, $C_2(s) = \frac{s+1}{0.3s+1}$ and $C_3(s) = -\frac{s+1}{-s+2.6}$.

A simple analysis of the non-switched system (true plant in feedback with each of the
controllers separately) shows that $C_1$ is stabilizing (yielding a non-minimum phase but stable closed loop) while $C_2$ and $C_3$ are destabilizing. Next, a simulation was performed of a switched system, where the algorithm A1 was used to select optimal controller, and the cost function to be minimized was chosen to be a combination of the instantaneous error and a weighted accumulated error:

$$J(t) = \tilde{e}_j^2(t) + \int_0^t e^{-\lambda(t-\tau)}\tilde{e}_j^2(\tau)d\tau, \quad j = 1, 2, 3$$  \hspace{1cm} (5.1)$$

Considering the control system setup in Figure 5.1, $\tilde{e}_j$ is the fictitious error of the $j^{th}$ controller, defined as

$$\tilde{e}_j = \tilde{y}_j - y$$  \hspace{1cm} (5.2)$$
and \( \tilde{y}_j = W_{ref} \tilde{r}_j \) and \( \tilde{r}_j = y + K_j^{-1}u \). This is the same cost function used in the multiple model switching adaptive control scheme [NB97], with \( \tilde{e}_j \) replaced by \( e_{I_j} \), the identification error of the \( j^{th} \) plant model. (For the special case of the candidate controllers designed based on the MRAC method, it was shown in [PS03] that \( e_{I_j} \) is equivalent to the control error and to the fictitious error (5.2)).

The simulations assume a band-limited white noise at the plant output and a unit-magnitude square reference signal. The stabilizing controller \( C_1 \) is initially placed in the loop, and the switching is allowed after 5 seconds. The forgetting factor \( \lambda \) is chosen to be 0.05. Figure 5.2 and Figure 5.3 show the cost dynamics and the reference and plants outputs, respectively.

The switching algorithm using cost function (5.1) discards the stabilizing controller initially placed into the loop and latches onto a destabilizing one, despite the evidence of instability found in the data. This is due to the lack of cost detectability of (5.1). Next, the simulation is performed of the same system, but using a ‘good’ cost function (one that satisfies the conditions of Theorem 3.2.1):

\[
V(K, z, t) = \max_{\tau \in [0, t]} \frac{||u||^2 + ||\tilde{e}_K||^2}{||\tilde{r}_K||^2 + \alpha}
\]

which is an \( L_2 \) gain type cost (factor \( \gamma ||K||^2 \) added for coerciveness is not necessary, since the set of candidate controllers is finite here). The corresponding simulation
Figure 5.2: Switching using cost function (5.1): Current values of the cost (5.1) for each controller.

Figure 5.3: Switching using cost function (5.1): Reference and plant outputs.
results are shown in Figures 5.4 and 5.5. The initial controller is chosen to be $C_3$ (a destabilizing one). The constant $\alpha$ is chosen to be 0.01.

The destabilizing controllers are kept out of the loop due to the properties of the cost function (cost detectability). For further comparison, the same simulation is repeated, with the destabilizing initial controller $C_3$, but the switching is allowed after 5 seconds (as in the first simulation run shown in Figure 5.2). It can be seen from Figure 5.6, that, despite the forced initial latching to a destabilizing controller, and therefore increased deviation of the output signal from its reference value, the algorithm quickly switches to a stabilizing controller, eventually driving the controlled output to its desired value.
Figure 5.5: Switching using cost function (5.3) and the switching algorithm A1. Reference and plant outputs.

Figure 5.6: Switching using cost function (5.3) and the switching algorithm A1, with the initial delay of 5 sec. Reference and plant outputs.
5.2 Comparison with the Multiple Model Based Switching Control of [NB97]

The case study that served as a source of inspiration for counterexamples in the beginning of the study that resulted in this thesis is found in the work of Narendra and Balakrishnan [NB97, Bal96]. In Chapter 3 of [Bal96], a comparison is drawn between the indirect MRAC adaptive control using switching among multiple models, and the indirect adaptive control using hysteresis switching algorithm of Morse [MMG92, Mor96]. It is stated that the choice of the performance index (cost function) used throughout [Bal96], namely:

\[
J_j(t) = \alpha e_j^2(t) + \beta \int_0^t e^{-\lambda(t-\tau)} e_j^2(\tau) d\tau
\] (5.4)

which incorporates both instantaneous and long term measures of accuracy, cannot be used in exactly the same form in the stability proof of the hysteresis switching lemma of [MMG92], since the identification errors \(e_j\) cannot be considered square integrable or even bounded, implying that the cost (5.4) does not satisfy the boundedness condition required for a stabilizing model. In [Bal96] a modification to the above cost is proposed in the form of:

\[
\bar{J}_j(t) = \int_0^t \frac{e_j^2(t)}{1 + \bar{\omega}^T(\tau)\bar{\omega}(\tau)} d\tau \quad & j = 1, ..., N
\] (5.5)

where \(\bar{\omega}\) is the on-line estimate of the vector \(\bar{\omega}^*\) consisting of the plant input, output, and sensitivity vectors, as is well known from the traditional MRAC theory for LTI
SISO plant [NA89]. Such a modified cost function satisfies the stability and finite time switching conditions of [MMG92]. The problem pointed out, however, was that this choice of the cost function is based strictly on the stability considerations, not performance, whereas the design parameters $\alpha, \beta, \lambda$ in (5.4) are claimed to provide flexibility in optimizing performance.

In [Bal96], superiority of the cost index choice (5.4) over (5.5) is advocated and a simulation is furnished showing a substantially better transient response using (5.4) rather than (5.5). Though it may seem that the former cost index results in a superior performance, an important issue was not accounted for: Are the conditions under which stability of the proposed control design is assured verifiable? This brings us to the essence of the problem we have approached in this thesis. We answer this question in the negative, since it is clear that one does not know a priori whether there exists a mismatch between the true plant and the proposed models. To provide better understanding, let us reconsider the problem presented in [Bal96] (Chapter 3, Section 3.3.3).

The LTI SISO plant to be controlled is assumed to have the transfer function (unknown to us) $G(s) = \frac{0.5}{s^2 + 0.05s + 1}$. The control objective is to track the output of a reference model $W_m = \frac{1}{s^2 + 1.4s + 1}$ to a square wave reference input with a unit amplitude and a period of 4 units of time. Two (fixed) candidate models are considered: $W_{F1}(s) = $
\( \frac{1}{s^2 + s - 1.5} \) and \( W_{F2}(s) = \frac{2}{s^2 + 2s + 1} \). MRC controllers are designed (off-line) for each of the two models, with the following parameter vectors: \( \theta^*_{F1} = [1, -0.4, -2.5, -0.6] \) and \( \theta^*_{F2} = [0.5, 0.6, -0.3, 0] \), where

\[
\begin{align*}
  u^*_{Fi} &= \theta^*_{Fi} \cdot \omega = \left[ k^* \quad \theta^*_{1i} \quad \theta^*_{0i} \quad \theta^*_{2i} \right] \cdot \\
  &= \begin{bmatrix} r \\ \omega_1 \\ y \\ \omega_2 \end{bmatrix},
\end{align*}
\]

is a perfect match control law for the model \( W_{Fi} \) resulting in perfect following of the reference command. The sensitivity vectors \( \omega_1, \omega_2 : \mathbb{R}_+ \to \mathbb{R}^{2n}, (n \text{ order of the unknown plant}) \) are defined, as in the standard MRAC problems, as:

\[
\begin{align*}
  \dot{\omega}_1 &= \Lambda \omega_1 + lu \\
  \dot{\omega}_2 &= \Lambda \omega_2 + ly
\end{align*}
\]

with \((\Lambda, l)\) an asymptotically stable, controllable pair.

The resulting control actions are then calculated as:

\[
\begin{align*}
  K_1 : & \quad u^*_{F1} = \frac{s + 1}{s + 1.4} r - \frac{2.5s + 3.1}{s + 1.4} y \\
  K_2 : & \quad u^*_{F2} = \frac{s + 1}{2(s + 0.4)} r - \frac{0.3(s + 1)}{s + 0.4} y
\end{align*}
\]

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A nonswitched analysis is then performed, with the real plant in feedback with each of the designed controllers separately, showing that $K_1$ is stabilizing, whereas $K_2$ is not (closed loop has a pair of RHP poles close to the imaginary axis). In the next simulation, a switching between these two controllers is performed using the performance index advocated in [Bal96, NB97]:

$$J_j(t) = e_j^2(t) + \int_0^t e_j^2(\tau) d\tau$$

The results of the simulation are shown in Figure 5.9 (cost) and Figure 5.10 (output). The switching scheme of [NB97] using rule (5.9) gives preference to the destabilizing controller $K_2$ since the parameters of its corresponding plant model are closer to...
Figure 5.8: True plant in feedback with controller $K_2$.

Figure 5.9: Switching using cost function (5.9). Cost function trajectory for controller $K_1$ and $K_2$. 

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Figure 5.10: Switching using cost function (5.9). Reference and plant output.

those of the real plant $W_p$. In [NB97], the authors strive to exclude this possibility by assuring sufficient density of the plant models, in order to increase the probability that the real plant falls within the robustness bounds of at least one candidate plant model, so that the corresponding robustly stabilizing controller of that model will also be robustly stabilizing for the real plant. While this idea is intuitively appealing, we should always assure that we do not deviate from a stabilizing and sufficiently well performing controller (when there exists one) and latch onto an even destabilizing one, as the example is this section demonstrates.
Now consider the switching with the cost function in (5.5) which satisfies the conditions of the hysteresis switching lemma of Morse et al. [MMG92]:

\[
\bar{J}_j(t) = \int_0^t \frac{e_j^2(t)}{1 + \bar{\omega}^T(\tau)\bar{\omega}(\tau)} d\tau \quad \text{for} \quad j = 1, \ldots, N \tag{5.10}
\]

Indeed, this type of a cost functional satisfies some of the properties required by our Theorem 3.2.3, namely it reaches a limit (possibly infinite) as \( t \to \infty \); at least one \( \bar{J}_j(t) \) (in this case \( \bar{J}_1(t) \)) is bounded, and the indirect controller \( C_1 \) assures stability for the real plant. A hysteresis algorithm is employed with a hysteresis constant \( \varepsilon > 0 \) to prevent chattering. This should assure that the switching stops at the model \( I_1 \) for which \( \bar{J}_1(t) \) is bounded, and so in turn assure stability. Let us look at the simulation results. Figure 5.11 shows the output of the plant, together with the reference model (desired) output. Apparently, the selector (cost) function of [MMG92] still opted for the destabilizing controller. It can be explained in terms of the cost (5.5) lacking the cost detectability property - that is, it employs a ratio of the error signal (expressed in terms of the system output signal) and a signal composed of the plant input and output (in measured and filtered forms) - which are again considered output signals form the standpoint of the overall closed loop switched system. Therefore, this form of the cost function does not detect instability. Costs of both controllers can be seen to be growing (Figure 5.12), but \( J_2 \) is growing more slowly than and is bounded above by \( J_1 \).

Finally, consider the switching using the cost function:
Figure 5.11: Switching using cost function (5.5). Reference and plant output.

Figure 5.12: Switching using cost function (5.5). Cost function for controller $C_1$ and $C_2$. 
Figure 5.13: Switching using cost function (5.11). Cost function for controller $C_1$ (dotted line) and $C_2$ (solid line)

\[ V(K, z, t) = \max_{\tau \in [0,t]} \left( \tilde{e}_j^2(\tau) + \int_0^\tau \tilde{e}_j^2(\sigma) d\sigma \right) \]

which conforms to the conditions of the Theorem 3.2.1. The simulation results are shown in Figures 5.13 and 5.14. Although $C_2$ is switched initially in the loop, the switching algorithm recognizes its destabilizing property and replaces it with $C_1$. 

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Figure 5.14: Switching using cost function (5.11). Reference and plant output.
Chapter 6

Conclusion

6.1 Summary

This thesis presents results of the study of the switching adaptive control of the highly uncertain systems. These are systems whose dynamics, parameters, or uncertainty models may be insufficiently known for a variety of reasons, including, for example, difficulty in obtaining accurate model estimates, changes in subsystem dynamics, component failures, external disturbances, time variation of plant parameters (slow but consistent time variation or infrequent large jumps) etc. Control of uncertain systems has traditionally been attempted using, on the one hand, robust control techniques (classical $H_{\infty}$ robust control and modern enhancements using linear matrix inequalities (LMI) conditions and integral quadratic constraints (IQC)) [SP96, CS88, ZDG96], whose proofs of robust stability and performance hinge upon the knowledge of sufficiently small uncertainty bounds around the nominal model. On the other hand, adaptive control techniques aim to further enhance robustness for larger uncertainties by introducing an outer adaptive loop that adjusts (tunes) controller parameters based on the observed data. Both streams have inherent limitations: robust control methods
are valid insofar as the proposed models match the actual plant and uncertainty bounds; adaptation in conventional continuous adaptive tuning may be very slow in comparison to the swiftly changing plant dynamics/parameters or rapidly evolving environment, thus yielding unacceptable performance or even instability in the practical terms. The fact that a single controller (fixed or adaptive) may not be able to cope with insufficiently known or changing plant was the primary reason that brought forth the notion of switching in the context of adaptive control. A wide variety of switching algorithms have been proposed in the past twenty years, with nearly all of them basing their stability/performance proofs to some extent on prior assumptions, which are invariably difficult to verify and rarely hold in practice.

This mismatch between the reality and prior assumptions is the central core of the problem we have addressed in this work. In the previous chapters, we have given a theoretical explanation of, and a solution to, the model-mismatch stability problem associated with a majority of adaptive control design techniques. The algorithm and the methodology of adaptive switching control proposed in this thesis are based on the theory of control law unfalsification according to which a reliable adaptive control law is synthesized.
6.2 Applications and Future Work

Adaptive and learning control techniques have a significant potential to enhance robustness of stability and performance of the systems operating under uncertain conditions. For example, adverse operating conditions, to which aircraft control systems are often subjected, introduce impacts and risks that are difficult to anticipate, calling for a reliable and prompt control action. The result proposed in this thesis belongs to the class of control paradigms that fully utilize information in the accumulated experimental data, and maximize robustness by introducing as few prior assumptions as is presently known, while at the same time converging quickly to a stabilizing solution, often within a fraction of an unstable plant’s largest unstable time constant. Thus, it forms a particularly attractive solution for the design and analysis of the fast adaptive fail-safe recovery systems for battle-damaged aircraft control systems, missile guidance systems, reconfigurable communication networks, precision pointing and tracking systems. Also, it is of interest to investigate a combination of switching control and the inherent switching present in some ‘overconstrained’ mechanical systems (distributed manipulation problems such as microelectromechanical systems) [Mur02].

Secondly, the proposed safe switching control may provide a valuable alternative solution for the control of a platoon of partially or fully automated vehicles. Recent investigation has suggested that for increased safety in highway platooning, adaptive cruise control (ACC) systems (partially automated systems, [IC93]) could be enhanced
with a cooperative action, which requires installation and reliable maintenance of communication links [SGH04]. To avoid this additional overhead, yet be robust to interference, false alarms and signal drop-outs, switching adaptive control may prove to be a viable solution.

Switching control in biological systems is of interest because robustness in such micro-scale systems is difficult to maintain. A single feedback controller is unlikely to be able to ensure, for example, sufficient robustness in signal transduction and immune regulation. Multi-level complexity of these systems renders many available control algorithms computationally intractable. It is of interest to investigate how a bio-system evolves over time and copes with multiple levels of uncertainty. Useful examples range from molecular biological control of metabolism to organ system interaction to ecological regulation. On a different note, recent investigation in [JL05] has explored bioactivity such as electromyographic activity of the spine during Network Spinal Analysis treatment. It appears that there exists a feedback loop between the central nervous system and spinal biomechanics that switches between different modes, thereby inducing sensory motor instability.

An important subject for future research is the application of the proposed safe adaptive control paradigm in the aforementioned and other situations. Computational solvability of the algorithm (e.g. polynomial-time type) will be investigated, particularly for the
case when the set of candidate controllers is continuously parametrized. Tractability
issues may depend to some extent on the compactness of the candidate controller set,
and on its representability as a finite union of convex sets. Tools from the theoretical
computer science and artificial intelligence concepts (such as machine learning,
[Mit97]) will be used to characterize and enhance levels of algorithm solvability.

On the theoretical side, it is of interest to further explore efficient ways to continuously
and adaptively generate new candidate controllers on the fly, enhancing the system with
an additional supervisory loop with a hypothesis generating role. The theory presented
in this work relies on the sole assumption that the adaptive control problem, posed as
optimization problem, is feasible, which means that the solution exists in the pool of
candidate controllers. To the best of our knowledge, this assumption underlies, implicit-
itly or explicitly, all other adaptive schemes, thus it is minimal. If it happens, however,
that this assumption does not hold (e.g. when one starts out with an initially sparse
set of controllers), it is needed to have a certain hypothesis generator that will create
new candidate controllers as the system evolves. This direction of further research is
underway.
Reference List


Appendix A

Finiteness of the ratio $\frac{\|\hat{r}_{KN}\|}{\|r\|}$

Finiteness of $\frac{\|\hat{r}_{KN}\|}{\|r\|}$, the ratio of the fictitious reference signal of the last switched controller and the actually applied reference signal after the last switching instant, has been addressed in [WS05] for the special case of the linear time-invariant minimum phase controllers. In particular, Lemma 3 in [WS05] demonstrates boundedness of the ratio $\frac{\|\hat{r}_{KN}\|}{\|r\|}$ based on the premise that the outputs of two identical stable systems (one generating $r$ and the other generating $\hat{r}_{KN}$), driven by the same inputs and initialized by different initial conditions, asymptotically tend to the same steady state value - a fact that is known to hold for linear time-invariant systems, but is not generally true in nonlinear systems. Here, the derivation of an upper bound is performed without these constraints on the controller structure. In the following lemma, the controllers $K$ are not subject to linearity or time-invariance constraints.

Lemma A.0.1 Consider the switching feedback adaptive control system $\Sigma$ (Figure 2.1) together with the switching algorithm $A_1$, where the uniformly bounded reference input $r$ and output $z = [u, y]$ are given. Suppose there are finitely many switches. Let $t_N$ and $K_N$ denote the last switching instant and the last switched controller, respectively. Suppose that $\forall K \in K$ the fictitious reference signals $\hat{r}_K(z, t)$ in the
set $\tilde{R}(K, z, t)$ have stable structure. Then, $\frac{||\tilde{r}_{K_N}||_r}{||r||_r} < \infty$ uniformly in time $\forall \tau \in [t_N, \infty)$.

**Proof.** By the assumption the are finitely many switches. Consider the control configuration in Figure A.1. The top branch generates the fictitious reference signal of the controller $K_N$. Its inputs are the measured data $(y, u)$, and its output is $\tilde{r}_{K_N}$. The output is generated by the fictitious reference signal generator for the controller $K_N$, which (if it exists) is denoted $K_N^{CLI}$. In the middle interconnection, the signal $u_N$, generated as the output of the last controller $K_N$ excited by the actual applied signal $r$ and the measured plant output $y$, is simply inverted by passing through the causal left inverse $K_N^{CLI}$. Finally, the bottom interconnection has the identical structure as the top interconnection (series connection of $\hat{K}_t$ and $K_N^{CLI}$), except that it should generate the actual reference signal $r$. To this end, another input to the bottom interconnection is added (denoted $\omega$), as shown in Figure A.1. This additional input $\omega$ can be thought of as a compensating (bias) signal, that accounts for the difference between the subsystems generating $r$ and $\tilde{r}_{K_N}$ before the time of the last switch. In particular, it can be shown (as seen in Figure A.1) that $\omega = P_{t_N}(u_N - u)$ (due to the fact that $u_N \equiv u, \forall t \geq t_N$).

Since $\omega$ is defined on a finite time interval, it has a finite extended $\mathcal{L}_2$-norm (finite energy). Now, following the notation in [Zam66], let us define a relation $K^{CLI}$ on $1$ a generalization to a relation instead of an operator is considered in [Zam66]; however, if $K^{CLI}$ is not an ‘into’ mapping, then its incremental gain, as seen further in text, is infinite; thus we will actually limit our attention to the operators.
\[ L_2, \text{ such that } \tilde{r} \text{ (the fictitious reference signal) is } K^{CLI} \text{-related to } [u, y, \omega], \text{ where } ([u, y, \omega], \tilde{r}) \text{ is a pair belonging to } K^{CLI}. \]

As stated, we consider controllers whose inverses have stable structure; i.e. whose relation \( K^{CLI} \) is bounded and continuous. Then, the incremental gain of \( K^{CLI} \), defined as:

\[
\tilde{g}(K^{CLI}) \doteq \sup ||K^{CLI}(x_1)_t - K^{CLI}(x_2)_t|| ||(x_1)_t - (x_2)_t||
\]

is finite (the supremum is taken over all \( x_1, x_2 \in L_2 \), all \( K^{CLI}(x_1), K^{CLI}(x_2) \in L_2 \), and all \( t \in \mathbb{R}_+ \) for which \((x_1)_t \neq (x_2)_t\)). Thus,
\[
||K^{CII}(x_1) - K^{CII}(x_2)||_e \leq \tilde{g}(K^{CII}) \cdot ||x_1 - x_2||_e,
\]

for all \(x_1, x_2 \in \mathcal{L}_2\) \hspace{1cm} (A.2)

where \(||x||_e = ||x|| \doteq \sqrt{\int_0^\infty |x(\tau)| d\tau}\) if \(x \in \mathcal{L}_2\); otherwise \(||x||_e = \infty\).

Now, choose \(x_1(t) \doteq (u(t), y(t), 0)\) and \(x_2(t) \doteq (u(t), y(t), \omega)\) where \(\omega = P_{t_N}(u_N - u)\) is the compensating bias signal for the fictitious reference signal of the last switched controller (as defined in the above figure); since it is defined on a finite time interval \([0, t_N]\), its induced \(\mathcal{L}_2\) norm is finite.

Then, (A.2) reduces to:

\[
||\tilde{r}_{K_N} - r||_e \leq \tilde{g}(K^{CII}_N) \cdot ||\omega||_e < \infty
\]

\(\Rightarrow ||\tilde{r}_{K_N} - r||_e = ||\tilde{r}_{K_N} - r||\)

\(\Rightarrow (\tilde{r}_{K_N} - r) \in \mathcal{L}_2\) \hspace{1cm} (A.3)

\(\Rightarrow ||\tilde{r}_{K_N}|| \leq (||r|| + (\tilde{g}(K^{CII}_N) \cdot ||\omega||))\)

(A.6)

and

\[
\frac{||\tilde{r}_{K_N}||}{||r||} \leq 1 + \frac{\tilde{g}(K^{CII}_N)||\omega||}{||r||}
\]

(A.7)
where the right-hand-side of (A.7) is finite, since \( \tilde{g}(K_N^{C_{LI}}) < \infty \), and \( \|\omega\| \leq \|\omega_{t_N}\| \) (since \( \omega(t) = 0 \ \forall t > t_N \)).